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Mathematical Reviews

Vol. 4, No. 5 MAY, 1943 Pages 125-152

FOUNDATIONS

✻**Cooley, John C. A Primer of Formal Logic.** Macmillan Company, New York, 1942. xi+378 pp. $3.00.

This is a clearly written elementary textbook on symbolic logic. The five principal topics it covers are the calculus of elementary propositions, the functional calculus, formal deductive systems, number systems, and classical logic. The notation used is similar to that of the Principia. The author has been influenced by Quine in his discussion of the truth-table method, the calculus of classes, and the theory of types, though in other respects his treatment of logic differs from Quine's. The chapters on deductive systems and number systems show the influence of Tarski. Mathematicians will be interested in the chapter devoted to number systems, which contains a postulate system for the real numbers in terms of Dedekind cuts, as well as a treatment of Peano's postulates for the integers and the Whitehead-Russell class definition of numbers. Being intended as an introductory text, the book contains no startling innovations. There is little or no mention of such things as many-valued logics, the Heyting and Lewis systems, Gödel numbers, lambda-conversion, or semantics. The discussion of Aristotelian logic and the syllogism is relegated to the last chapter. The author's lucid style and the many explanations, illustrations and exercises should make this a very successful textbook. *O. Frink* (State College, Pa.).

Fitch, Frederic B. A basic logic. J. Symbolic Logic 7, 105–114 (1942).

This paper proposes a system F of logic similar to systems given by Church, Curry and Rosser. "Class" is used synonymously with property; a class of propositions whose members may be recursively defined is a "system." A class X is "definable" in a system L if every true proposition and no false proposition of the form "so and so is a member of X" is in L. For the case where X is a system the author presents evidence supporting his belief that (1) his system F is "basic" in the sense that every system of logic is definable in F, and (2) propositions not definable in F are not provable in any system. The decision problem is shown to be unsolvable for F. The system F is given not explicitly but as a usual calculus. *A. L. Foster* (Berkeley, Calif.).

Curry, Haskell B. The inconsistency of certain formal logics. J. Symbolic Logic 7, 115–117 (1942).

In a previous paper [Trans. Amer. Math. Soc. **50**, 454–516 (1941); these Rev. **3**, 129] the author simplified the method, due to Kleene and Rosser, of deriving the Richard paradox in certain formal logics. The present note describes a much simpler method of deriving, under weaker hypotheses, contradictions based on the Russell and Epimenides paradoxes. The derivation of the Russell paradox is particularly simple, not requiring even the machinery of Gödel numbers. In both cases it is assumed merely that the system is combinatorially complete in the sense of the previous paper, and that there is an implication operation $\supset$ of universal applicability with the properties: (I) $M \supset M$; (II) if $M \supset (M \supset N)$, then $M \supset N$; and (III) if M, and $M \supset N$, then N. [The statement of property (I) contains a misprint.] The author points out that his previous assumption (not made by Kleene and Rosser) of the existence of the constancy combinator K can be omitted if (I), (II) and (III) are assumed. It is indicated that the root of the contradictions lies in the assumptions concerning implication, and not in the hypothesis of combinatorial completeness. *O. Frink.*

Thompson, Samuel M. Syllogistic logic in linear notation. Philos. Sci. 9, 362–366 (1942). [MF 7391]

This paper describes a rapid method of determining all the conclusions which follow, according to the rules of the traditional syllogism, from any number of related premises in the A, E, I, O forms. This is accomplished by means of a notation which requires all terms occurring in the premises to be arranged in a geometrical pattern on the page. The letters a, i and o are then placed between the terms in a certain way, corresponding to the relations between them which the premises assert. The conclusions may then be read off from the resulting diagram by means of four rules of a geometrical nature which the author derives. The four disputed moods of the syllogism are accepted as valid without modification. *O. Frink* (State College, Pa.).

Berkeley, Edmund C. Conditions affecting the application of symbolic logic. J. Symbolic Logic 7, 160–168 (1942).

The author argues that symbolic logic, with certain changes in symbolism, content and manner of presentation, can be widely applied in such fields as business, engineering and the biological and social sciences, and that such application will promote the rapid progress of symbolic logic itself. For example, symbolic logic has already been used to remove loopholes in contracts, and to simplify electrical circuits, the design of complicated machinery and complex sets of rules and regulations. In the natural sciences it lends itself to the exact definition of nonnumerical concepts too complicated to be conveniently described in ordinary language. For further applications, the subject should, on the one hand, be enriched with added symbols and relations, while, on the other hand, its manner of presentation must be simplified and popularized. *O. Frink* (State College, Pa.).

Pankajam, S. On the formal structure of the propositional calculus. II. J. Indian Math. Soc. (N.S.) 6, 51–62, 102 (1942). [MF 7683]

[The first part appeared in the same J. (N.S.) **5**, 49–61 (1941); these Rev. **3**, 130]. In part I of this paper [loc. cit.] the author applied lattice-theoretic methods to the Heyting intuitionistic logic. She identified the Heyting negation a' of an element a in a distributive lattice with the product-

complement of a (that is, the maximal element b such that $a \cap b = 0$), and the Heyting implication relation with lattice inclusion. In the present installment she introduces the lattice-theoretic analogue of the Heyting implication operation $a \supset b$. This is called the product-complement of a relative to b, or the quotient of b by a, and is denoted by (b,a) or $b|a$. It has the defining property: $t \cap a \leqq b$ if and only if $t \leqq (b,a)$. It is, of course, an assumption concerning the lattice that such a quotient always exists. The author acknowledges that this definition of implication is equivalent to the lattice-theoretic definition previously given by Garrett Birkhoff [Lattice Theory, Amer. Math. Soc. Colloquium Publ., v. 25, New York, 1940, pp. 128–130; these Rev. 1, 325]. Her discussion of this operation is somewhat different from that of Birkhoff. She proves ten theorems about it, and shows that a distributive lattice in which relative product-complements always exist is equivalent to a Heyting calculus, and conversely. *O. Frink.*

Dexter, Glenn Edward. The calculus of non-contradiction. Amer. J. Math. 65, 171–178 (1943). [MF 7784]

This paper describes a method of formulating the calculus of elementary propositions in terms of a single operation, subject to a single assumption. The operation, which is applicable to any finite number of elements, is denoted by a horizontal bar placed over the terms to which it applies, and the resulting formula is called a parenthesis. The parenthesis $\overline{pqr}$, for example, is interpreted as the proposition that at least one of the three propositions pq, q, r is false. The single assumption is that, if P is a parenthesis such that if we had P we would have a contradiction, then we have every proposition which occurs in P. A collection of propositions is said to contain a contradiction if it contains a parenthesis P and also every proposition which occurs in P. By way of illustration the author works out the proofs of twenty logical theorems in his notation. These include the axioms and rule of inference of Principia Mathematica. Two rules of cancellation are derived which may be used to shorten proofs. *O. Frink* (State College, Pa.).

Kleene, S. C. Recursive predicates and quantifiers. Trans. Amer. Math. Soc. 53, 41–73 (1943). [MF 7824]

This paper gives a unified view of various incompleteness theorems, including those of Church and Gödel, by deriving them as consequences of a central non-equivalence theorem (the author's theorem II). This states that, for each of a certain list of canonical forms, there exists an elementary numerical predicate expressible in that form but not in any other form of the list with the same or a smaller number of quantifiers. For example, the theorem asserts the existence of an elementary predicate expressible in the canonical form $(x)E(y)R(a, x, y)$, but not in the forms $E(x)(y)R(a, x, y)$ or $(x)R(a, x)$, where R is general recursive. By an elementary predicate is meant a property of integers expressible in terms of general recursive predicates and the logical connectives and quantifiers. A slightly weaker non-equivalence theorem is provable if only intuitionistically valid steps are allowed. The proof, which is effective, involves the Cantor diagonal process applied to a Gödel enumeration of predicates of a given form. It follows that there exist one-quantifier predicates for which the decision problem is unsolvable (Church's theorem), and that there exist one-quantifier predicates for which there is no complete formal deductive theory (Gödel's theorem). An incompleteness theorem for ordinal logics in the sense of Turing is also derived. It is shown that there exists a proposition of the form $(x)E(y)(z)R(x, y, z)$ which is classically true but not constructively provable, since no general recursive $\phi(x)$ exists for which $(x)(z)R(x, \phi(x), z)$ holds. The author proposes that the intuitionistic meaning of $(x)E(y)A(x, y)$ be identified with the existence of a general recursive $\phi(x)$ such that $(x)A(x, \phi(x))$ holds, in which case the classical and intuitionistic theories are demonstrably divergent. Finally an example is given of a non-elementary numerical predicate.

The paper also provides a complete and self-contained exposition of the theory of primitive recursive, general recursive, and partial recursive functions and predicates. In particular it is shown that, classically, a numerical predicate is general recursive if and only if it is expressible in both one-quantifier forms, and that in the canonical forms with at least one quantifier, the function R may without loss of generality be restricted to be primitive recursive instead of general recursive. *O. Frink* (State College, Pa.).

Newman, M. H. A. On theories with a combinatorial definition of "equivalence." Ann. of Math. (2) 43, 223–243 (1942). [MF 6460]

The author considers combinatorial theories in which an element can be transformed into an equivalent element by a series of moves or elementary operations, for instance, the insertion or removal of pairs xx^{-1} and $x^{-1}x$ in the elements of a free group, or in combinatorial topology the breaking of an edge by insertion of a vertex or conversely. Let us denote such a move from a to b by $a\mu b$. Often the moves fall into a positive class and its converse or negative class so that $a\mu b$ defines a partial order. The main problem is to determine when two elements connected by a path of moves have a common endform or element contained in both which can be reached by positive moves from either one. This is proved to be the case when the condition holds: When $a\mu x$ and $a\mu y$ then there exists a b such that $x\mu b$ and $y\mu b$. Under these circumstances it follows also that any two descending chains to the endform have the same length, an extension of the theorem of Jordan-Hölder to partially ordered sets. Some results of a similar nature are derived when the partial order is composed of two types of moves $x\lambda y$ and $x\rho y$.

A more complicated theory is developed by means of the concept of derivate sets. Let $\xi = x\mu z$ and $\eta = x\mu y$ be moves from the element x. To each η there is associated a derivate set $\eta|\xi$ consisting of certain moves from z, and similarly to every set E_x of moves from x the set $E_x|\xi$ of all derivate sets of the moves in E_x. By proceeding along any descending path Π from x one defines successively the derivate set $E_x|\Pi$ with respect to the endpoint of the chain. A development of E_x is a path proceeding in each case within the derivate sets and a development is complete when it ends. Certain axiomatic conditions are imposed which make it possible to prove that any two complete developments have the same endpoint.

The main application of the theory is to a proof of the normal form theorem of Church and Rosser in the conversion calculus. *O. Ore* (New Haven, Conn.).

Chandrasekharan, K. The logic of intuitionistic mathematics. Math. Student 9, 143–154 (1941). [MF 7030]

This is a short introduction to the intuitionistic logic of Heyting and the intuitionist set theory of Brouwer. The principal features of the Heyting logic are presented, usually

without proof. Some of this material is new, but most of it is well known. The lattice theory treatment of Heyting's logic due to S. Pankajam [J. Indian Math. Soc. (N.S.) **5**, 49–61 (1941); these Rev. **3**, 130] is mentioned, but not that of Garrett Birkhoff [Lattice Theory, American Mathematical Society Colloquium Publ., vol. 25, New York, 1940, pp. 128–130; these Rev. **1**, 325]. The description of Brouwer's set theory is confined mostly to definitions and examples. *O. Frink* (State College, Pa.).

Levi, Beppo. Methods of attack in logic. Univ. Nac. Tucumán. Revista A. **3**, 13–78 (1942). (Spanish) [MF 8146]

Greenwood, Thomas. La méthode axiomatique en géométrie. Rev. Trimest. Canad. **28**, 353–379 (1942). [MF 7637]

The paper presents a philosophical discussion of the axiomatic method with illustrations from the foundations of projective, elliptic, descriptive, absolute and Euclidean geometry. *L. M. Blumenthal* (Columbia, Mo.).

Black, Max. Conventionalism in geometry and the interpretation of necessary statements. Philos. Sci. **9**, 335–349 (1942). [MF 7390]

Birkhoff, G. D. Is a mathematical theory of aesthetics possible? Revista Ci., Lima **44**, 241–243 (1942). (Spanish) [MF 8034]

Birkhoff, G. D. Is a mathematical approach to ethics possible? Revista Ci., Lima **44**, 245–248 (1942). (Spanish) [MF 8035]

Birkhoff, G. D. The principle of sufficient reason. Revista Ci., Lima **44**, 249–251 (1942). (Spanish) [MF 8036]

Birkhoff, G. D. Modern logic and mathematics. Revista Ci., Lima **44**, 251–252 (1942). (Spanish) [MF 8037]

Summaries (prepared from the author's notes) of lectures given by the author at the Universidad Nacional Mayor de San Marcos, in Lima, April and May, 1942. For more extensive presentations of the author's theories, cf. Rice Inst. Pamphlet **19**, no. 3 (1932) [aesthetics]; **28**, no. 1 (1941) [ethics and sufficient reason].

ALGEBRA

Narasimha Murti, V. A problem in combinations. Math. Student **10**, 85–86 (1942). [MF 7958]

The problem under consideration is that of selecting from m soldiers guards for m days in such a way that any two soldiers are companions on two days only. This problem was considered recently by Gupta [Math. Student **8**, 131–132 (1940); these Rev. **2**, 341], who stated that if n is the number of soldiers in each guard then the problem is possible only if $m-1=n(n-1)/2$. In reviewing Gupta's paper, the present reviewer appears to have been in some doubt of this fact. He now takes this opportunity to state that this condition is quite correct and follows at once if one enumerates in two ways the total number of salutations given by the soldiers to one another during the whole number m of days. Gupta gave solutions of the problem for $n=3, 4, 5, 6$, the first three being "elegant" solutions in the following sense. If we number the soldiers from 0 to $m-1$, then an elegant solution is one in which the numbers of the men in any day's guard are obtained by simply increasing by unity (and reducing modulo m if necessary) the numbers of the men in the previous guard, so that the whole scheme is determined and can be described by the numbers in the first day's guard. The elegant solutions of Gupta for $n=3$, 4 and 5 are thus determined by 0, 1, 2; 0, 1, 2, 4; and 0, 1, 2, 4, 7. The purpose of the present paper is to point out that for $n=6$ ($m=16$) no elegant solution exists. Gupta has conjectured that a solution (elegant or otherwise) exists for each n. *D. H. Lehmer* (Berkeley, Calif.).

Fisher, R. A. The theory of confounding in factorial experiments in relation to the theory of groups. Ann. Eugenics **11**, 341–353 (1942). [MF 7973]

Let $\alpha, \beta, \cdots$ denote the n generators of the Abelian group of order $m=2^n$ and type $(1, \cdots, 1)$, so that the m elements of the group are $1, \alpha, \beta, \alpha\beta, \cdots$. Let Latin letters $A, B, \cdots$ be associated with the $m-1$ elements other than the identity, and let each generator be represented as a formal product of the Latin letters associated with the $m/2$ elements which involve that generator. If the Latin letters (like the Greek letters) are then regarded as permutable involutory operations, every element of the group (save the identity) will appear as a product of $m/2$ Latin letters (namely, the Latin letters associated with the $m/2$ elements which have an odd number of Greek letters in common with the given element). For instance, if the letters A, B, C, D, E, F, G are associated with the respective elements α, β, γ, $\beta\gamma$, $\alpha\gamma$, $\alpha\beta$, $\alpha\beta\gamma$ of the Abelian group of order 8, we write $\alpha=AEFG$, $\beta=BDFG$, $\gamma=CDEG$ and deduce $\beta\gamma=BCEF$, $\alpha\gamma=ACDF$, $\alpha\beta=ABDE$, $\alpha\beta\gamma=ABCG$. The Abelian group of order $m=2^n$ and type (1^n) is thus exhibited as a subgroup ("the intrablock group") in the Abelian group of order 2^{m-1} and type (1^{m-1}) generated by the $m-1$ Latin letters.

The consequent scheme of $m-1$ sets of $m/2$ letters is particularly symmetrical, as each of the $m-1$ letters appears in $m/2$ of the sets, and each of the $\binom{m-1}{2}$ pairs appears in $m/4$ of the sets. ("With only 2^n treatments in each block, we may test 2^n-1 factors and all their interactions without confounding any interaction of less than three factors.") Moreover, if $n>2$, certain triads of letters appear in $m/8$ of the sets, but there remain $\binom{m-1}{2}/3$ "confounded" triads which do not appear in any. These (or a suitable subset of $m-n-1$) generate an Abelian group of order 2^{m-n-1} ("the group of confounded interactions"), whose elements each have an even number of (Latin) letters in common with every element of the group of order m. The

$$N=(m-1)(m-2)(m-2^2)\cdots(m-2^{n-1})$$

automorphisms of the Abelian group of order m all appear as permutations of the $m-1$ letters. Hence the number of distinct representations of the above kind is $(m-1)!/N$.

If the elements of the Abelian group of order 2^{m-1} are represented in the natural manner by the vertices of an $(m-1)$-dimensional hyper-cube, the subgroup of order m appears as a regular simplex inscribed in the hyper-cube. It is known [see Coxeter, J. Math. Phys. Mass. Inst. Tech. **12**, 334–345 (1933), in particular, p. 341] that a symmetrical set of $2^{m-1}(m-1)!/mN$ simplexes can be so inscribed, and that each vertex of the hyper-cube (for example, the vertex representing the identity) belongs to $(m-1)!/N$ of them. When $m=4$, so that $N=6$, this is the familiar figure of two tetrahedra inscribed in a cube.

By deleting r of the $m-1$ letters (or putting those letters equal to 1), the author exhibits the Abelian group of order m as a subgroup in the Abelian group of order 2^{m-1-r} generated by the remaining letters. Details are given for the cases when $m=16$ and $0 \leqq r < 8$. Table (a), on page 346, can be regarded as an eight-dimensional cross-polytope (the analogue of the octahedron) inscribed in the eight-dimensional hyper-cube, and table (v), on page 353, as a fifteen-dimensional simplex inscribed in a fifteen-dimensional hyper-cube. [In the latter table, *ABDGJKMN* is a misprint for *ABDGJKMP*.] *H. S. M. Coxeter* (Toronto, Ont.).

Bose, R. C. A note on two series of balanced incomplete block designs. Bull. Calcutta Math. Soc. **34**, 129–130 (1942). [MF 7577]

The purpose of this note is to prove that, if x is a primitive element of the field $GF(p^n)$, where $p^n=4t+1$, then there exists an odd α such that $(x^\alpha+1)/(x^\alpha-1)$ is an odd power of x. The author considers the involution $yz-y-z-1=0$, which distributes the elements $x, x^2, \cdots, x^{2t-1}, x^{2t+1}, \cdots, x^{4t-1}$ into $2t-1$ pairs (y, z), and remarks that we have only to show that at least one pair consists of two odd powers of x. The work on the second page is superfluous, as it would have been sufficient to observe that the $4t-2$ elements just named consist of $2t-2$ even powers and $2t$ odd powers. The author uses this result to establish the existence of a balanced incomplete block design [cf. Ann. Eugenics **9**, 353–399 (1939), in particular, pp. 385, 388; these Rev. **1**, 199] with $\lambda=1$, $k=4$ or 5 and $r=kt+1$, whenever $4t+1$ is a power of a prime. *H. S. M. Coxeter* (Toronto, Ont.).

Kesava Menon, P. The evaluation of certain determinants. Math. Student **10**, 75–79 (1942). [MF 7955]

Evaluation of some interesting determinants such as $D_n((x_r+y_s)^{-1})$, which was first given by Cauchy in 1841, and later by several authors. No literature is mentioned. *O. Szász* (Cincinnati, Ohio).

Bruck, Richard H. and Wade, T. L. Bisymmetric tensor algebra. I. Amer. J. Math. **64**, 725–733 (1942). [MF 7174]

A bisymmetric $2p$-tensor is a tensor $A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_p} = A^{(i)}_{(j)}$, which is unchanged if both the i and the j are subjected to the same arbitrary permutation. Determinant, adjoint and inverse of such a tensor are defined with the aid of a generalized Kronecker delta:

$$\delta^{(i_1)(i_2)\cdots(i_N)}_{(j_1)(j_2)\cdots(j_N)} = \begin{vmatrix} \delta^{(i_1)}_{(j_1)} & \delta^{(i_1)}_{(j_2)} & \cdots & \delta^{(i_1)}_{(j_N)} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \delta^{(i_N)}_{(j_N)} \end{vmatrix}, \qquad N=n^p,$$

where

$$\delta^{(i)}_{(j)} = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\cdots\delta^{i_p}_{j_p}.$$

Then the determinant of $A^{(i)}_{(j)}$ is defined by

$$|A| = (N!)^{-1}\delta^{(i_1)(i_2)\cdots(i_N)}_{(j_1)(j_2)\cdots(j_N)} A^{(i_1)}_{(j_1)} A^{(i_2)}_{(j_2)} \cdots A^{(j_N)}_{(i_N)}.$$

D. J. Struik (Cambridge, Mass.).

Bruck, Richard H. and Wade, T. L. Bisymmetric tensor algebra. II. Amer. J. Math. **64**, 734–752 (1942). [MF 7175]

The set $\mathfrak{L}$ of all linear terms of the unit tensor $\delta^{(i)}_{(j)}$ and its isomers forms a tensor algebra. Any absolute numerical tensor is, for some p, a member of this algebra. For a given p the algebra $\mathfrak{L}$ comprises the commutator algebra of the algebra of all bisymmetric tensors, and conversely. Let $C^{(i)}_{(j)}$ be a numerical idempotent tensor; that is, let $C^{(i)}_{(m)}C^{(m)}_{(j)} = C^{(i)}_{(j)}$. In terms of such a tensor we can define a subalgebra of the total $2p$-tensor algebra, in which $C^{(i)}_{(j)}$ is the unit. Then come the invariant subalgebras $\mathfrak{S}_{[a]}$ of the algebra $\mathfrak{B}$ of bisymmetric tensors, defined in terms of the "immanent" tensor ${}_{(a)}I^{(i)}_{(j)}$, which is the idempotent numerical tensor corresponding to the irreducible representation $[a]$ of the symmetric group on p letters [T. L. Wade, Amer. J. Math. **63**, 645–657 (1941); these Rev. **3**, 19]. A method is given for calculating the rank of any immanent and to construct determinant and inverse of any tensor belonging to $\mathfrak{S}_{[a]}$. Such an $\mathfrak{S}_{[a]}$ is the direct sum of f_a equivalent tensor algebras, where f_a is the characteristic corresponding to class (1^p). These algebras $\mathfrak{S}_{[a]}$ are the only minimal invariant subalgebras of $\mathfrak{B}$. Some applications to algebras associated with a Young diagram follow. *D. J. Struik* (Cambridge, Mass.).

Robinson, G. de B. Note on a paper by R. H. Bruck and T. L. Wade. Amer. J. Math. **64**, 753 (1942). [MF 7176]

In the paper reviewed above, the rank r_a of an immanent tensor ${}_{(a)}I^{(i)}_{(j)}$ is found. This number is expressed here in terms of Young's substitutional analysis. *D. J. Struik.*

Abstract Algebra

Ore, Oystein. Theory of equivalence relations. Duke Math. J. **9**, 573–627 (1942). [MF 7337]

This paper presents an extended treatment of equivalence relations on an arbitrary set S. There is a one-one correspondence between tha equivalence relations and the partitions of S. A partition determines a complete field of subsets of S, and conversely. The system of all partitions of S forms a complete lattice (structure). There is a lattice isomorphism between the lattice of all partitions of S and the lattice of complete fields over S. The lattice of partitions of S is neither modular nor distributive. However, necessary and sufficient conditions are established that the Dedekind (modular) law holds for two arbitrary partitions and a third partition which satisfies certain conditions relative to the other two partitions, and also that the distributive law holds similarly. There are sections on the law of isomorphism and chain refinements, on commuting equivalence relations and on distributive decompositions in the lattice of all partitions of S. There is a discussion of various kinds of complementation. Many of the results in the preceding theory are interpreted as properties of a group G and its subgroups, using the coset partitions.

A correspondence of S to a subset of S establishes a partition of S, each of whose blocks consists of the elements of S which correspond to the same element of the subset. Conversely, an arbitrary partition of S can be established by an appropriately defined correspondence of S to a subset. Various properties of the lattice of all partitions of S are studied by such correspondences. The group of all automorphisms of an equivalence relation is explicitly determined. There is a representation of an arbitrary lattice by such correspondences. This representation is a lattice isomorphism if the original lattice is distributive. There is an axiomatic characterization, in geometric form, of the lattice of partitions of S. The group of automorphisms of

this lattice is the symmetric group on S. Homomorphisms of lattices, in particular, of the lattice of partitions of S, are analyzed. *L. W. Griffiths* (Evanston, Ill.).

Whitman, Philip M. Splittings of a lattice. Amer. J. Math. **65**, 179–196 (1943). [MF 7785]

By a splitting is meant a partition which divides a lattice L into an ideal P and a dual ideal D; P and D are called splinters; the elements of P covered by elements of D, respectively the elements of D covering elements of P, the edges of the splinters. The author shows that these edges are isomorphic partially ordered sets; he also shows (giving necessary and sufficient conditions) how to reconstruct L from P and D. For L to be modular (distributive) it is necessary and sufficient that (i) the same hold for P,D, and (ii) the edges of the latter be interval sublattices. This is all assuming a finite chain condition or less; various other results are obtained. *G. Birkhoff* (Cambridge, Mass.).

Krishnan, V. S. The problem of the last-residue-class in the distributive lattice. Proc. Indian Acad. Sci., Sect. A. **16**, 176–190 (1942). [MF 7792]

Let L be a distributive lattice with units $(0, 1)$. A subset K of L is called a μ-ideal if $a\epsilon K$, $b\epsilon K$, $x\epsilon L$ imply $a+b\epsilon K$, $xa\epsilon K$, and an α-ideal if the dual conditions hold. Two elements x and y of L are said to be congruent modulo K if there exist elements a and b of K such that $x+a=y+b$, or $xa=yb$, and the last residue-class of K is the set of $x\equiv 1$, or $x\equiv 0$, modulo K, according as K is a μ- or α-ideal, respectively. The last residue-class of a μ-ideal is an α-ideal, and vice versa. The author discusses the problems: (1) to characterize the last residue-class of an ideal by means of ideal operations on K; (2) to determine conditions under which an ideal of L is the last residue-class of another ideal; (3) to determine the ideals which are identical with the last residue-class of their last residue-class. He obtains partial solutions for the general L and complete solutions for Boolean algebras. Certain new concepts are introduced.

R. Hull (Vancouver, B. C.).

Artin, Emil and Whaples, George. The theory of simple rings. Amer. J. Math. **65**, 87–107 (1943). [MF 7779]

The paper presents an exposition of the theory of simple rings with minimal condition on left ideals; besides known results (such as Wedderburn's structure theorem and Brauer's theorem on the direct product of an algebra and its reciprocal), there are several generalizations of theorems previously proved only for algebras of finite order. Examples are (1) an isomorphism of two simple subalgebras of a simple ring R can be extended to an inner automorphism of R; (2) simple subalgebras of R occur in pairs each of which is the commutator of the other; (3) if A is a simple subalgebra of R, then any automorphism of R that leaves the commutator of A elementwise fixed is an inner automorphism by an element of A. The authors make consistent use of functions of the form $\sum a_i x b_i$ ("analytic" functions) and of direct products, for which they give a definition independent of basis. *I. Kaplansky* (Ann Arbor, Mich.).

Grundy, P. M. A generalization of additive ideal theory. Proc. Cambridge Philos. Soc. **38**, 241–279 (1942). [MF 7810]

This paper deals with the decomposition theory of modules $\mathfrak{M}=\{\alpha, \beta, \cdots, \mathbf{O}\}$ admitting the elements $a, b, \cdots$ of a commutative ring $\mathfrak{R}$ as linear operators. In the sequel it is assumed that $1\alpha=\alpha$ for the unity 1 of $\mathfrak{R}$. Moreover, each $\mathfrak{R}$-permissible submodule $\mathfrak{m}$ of $\mathfrak{M}$ is to have a finite set of generators $\mu_1, \cdots, \mu_n$; that is, for each μ in $\mathfrak{m}$ there is a representation $\mu=\sum_{j=1}^n a_j\mu_j$, where the a_j are either integers or elements of $\mathfrak{R}$. The author shows that the results of the Lasker-Noether ideal theory can be extended to modules. Furthermore, the discussion of contracted and extended ideals and its natural sequel, the theory of quotient rings, can be generalized. An important tool for the proofs, as in the additive ideal theory, is furnished by (i) the ideal quotient $\mathfrak{m}:\mathfrak{n}=$ all a with $a\mathfrak{n}\subseteqq\mathfrak{m}$, and (ii) the module quotient $\mathfrak{m}:\mathfrak{a}=$ all α with $\mathfrak{a}\alpha\subseteqq\mathfrak{m}$. There is then a 1-1 correspondence between the annihilators $\mathfrak{m}, \mathfrak{a}$ given by $\mathbf{O}:\mathfrak{m}=\mathfrak{a}$ and $\mathbf{O}:\mathfrak{a}=\mathfrak{m}$. The following concepts enable the author to translate results of the ideal theory to new theorems for modules. They are (i) the shadow $\mathfrak{m} : \mathfrak{M}$ of a module $\mathfrak{m}$ and (ii) the e-modules $\mathfrak{m}$ for which there exists at least one ideal $\mathfrak{a}$ with $\mathfrak{a}\mathfrak{M}=\mathfrak{m}$. There is a 1-1 pairing of shadows and e-modules given by $\mathfrak{m}:\mathfrak{M}=\mathfrak{a}$ and $\mathfrak{a}\mathfrak{M}=\mathfrak{m}$. A module $\mathfrak{m}$ is termed primary (prime) if $b\alpha\subseteqq\mathfrak{m}$ and $\alpha\mathfrak{R}\nsubseteqq\mathfrak{m}$ imply $b^r\epsilon\mathfrak{m}:\mathfrak{M}$ $(b\epsilon\mathfrak{m}:\mathfrak{M})$. It follows that the shadow of a primary (prime) module is a primary (prime) ideal. Finally, the prime ideal $\mathfrak{p}$ of the shadow $\mathfrak{m}:\mathfrak{M}$ is termed its radical.

With these preparations it is easy to prove the general reduction theorem of modules: (i) each module has a Lasker reduction $\mathfrak{m}=\mathfrak{Q}_1\cap\cdots\cap\mathfrak{Q}_n$, where the $\mathfrak{Q}_j$ are primary modules with distinct radicals and no $\mathfrak{Q}_j$ is redundant; (ii) the radicals $\mathfrak{p}_j$ of the $\mathfrak{Q}_j$ and their number n are the same for all possible Lasker decompositions of $\mathfrak{m}$; (iii) all Lasker decompositions have the same isolated components. Here a component is called isolated if its prime ideals are an isolated subset of those belonging to $\mathfrak{m}$; that is, it is a subset of prime ideals which includes every $\mathfrak{p}_i$ that is a subideal of a member of that set. Moreover, the isolated prime ideals of the shadow $\mathfrak{m}:\mathfrak{M}$ are the same as those of $\mathfrak{m}$; the distinct isolated primary components of $\mathfrak{m}:\mathfrak{M}$ are the shadows of the distinct isolated primary components of $\mathfrak{m}$. These results are followed by a detailed discussion of the direct decompositions of a module; it is essentially an extension of the Chinese remainder theorem. The paper closes with an investigation of the Loewy series belonging to a module whose radical is maximal. [For comparison, see Snapper, Trans. Amer. Math. Soc. **52**, 257–264 (1942); these Rev. **4**, 13.] *O. F. G. Schilling*.

Forsythe, Alexandra. Divisors of zero in polynomial rings. Amer. Math. Monthly **50**, 7–8 (1943). [MF 7941]

The author gives an alternative proof of the recent theorem of McCoy [Amer. Math. Monthly **49**, 286–295 (1942); these Rev. **3**, 262] that, if $f(x)$ is a divisor of zero in $R[x]$, there exists a non-zero element c of R such that $c\cdot f(x)=0$. *C. C. MacDuffee* (New York, N. Y.).

Bell, E. T. Polynomials on a finite discrete range. Duke Math. J. **10**, 33–47 (1943). [MF 8099]

The author defines $A_n^{(s)}$ to be the set of all functions of s variables where the variables and the values of the functions are both confined to the range $0, 1, \cdots, n-1$; $A_n^{(1)}$ can be identified with the propositional calculus for an n-valued truth system, and in general $A_n^{(s)}$ is a special case of a Post algebra as defined by P. C. Rosenbloom [Amer. J. Math. **64**, 167–188 (1942); these Rev. **3**, 262]. A systematic notation for the elements of $A_n^{(s)}$ is developed and several unique normal forms are exhibited. While $A_n^{(s)}$ is

not closed under multiplication for $n>2$, it is closed under composition; and in conclusion the author shows how one might use these functions under composition in finding algebraic systems possessing specified properties (e.g. commutativity, associativity, etc.). *I. Kaplansky.*

Kaloujnine, Léo. Sur la théorie de Galois des corps non galoisiens séparables. C. R. Acad. Sci. Paris **214**, 597–599 (1942). [MF 7891]

Let $K=k(\theta)$ be a separable algebraic extension of degree n of the field k, and let $\mathfrak{R}$ be the regular representation of K, regarded as a hypercomplex system over k. Let $\mathfrak{F}_1, \mathfrak{F}_2, \cdots, \mathfrak{F}_r$ be the inequivalent irreducible constituents of $\mathfrak{R}$ into which $\mathfrak{R}$ splits when k is extended to K. Let ρ_i be the isomorphic mapping of K on $\mathfrak{F}_i$ $(i=1, \cdots, r)$. At least one of the $\mathfrak{F}_i$ is of degree 1, corresponding to the identity automorphism of $K: \theta\to\theta$, but, unless K is normal over k, some $\mathfrak{F}_i$ will be of degree exceeding 1. The author calls the mappings ρ_i hypermorphisms of K relative to k and defines the product $\rho_1\rho_2$ of any two of them as follows. In the matrices of $\mathfrak{F}_2$ substitute for the coefficients, which are elements of K, the matrices to which they correspond in the mapping ρ_1. There results a representation of K which contains certain of the $\mathfrak{F}_i$ as its irreducible constituents, say, $\mathfrak{F}_{i_1}, \mathfrak{F}_{i_2}, \cdots$. Then $\rho_1\rho_2$ is defined to be the set $\rho_{i_1}, \rho_{i_2}, \cdots$. The author states that under this composition the hypermorphisms form a hypergroup (multigroup) and that the Galois correspondence between subfields and subhypergroups can be directly proved without reference to a normal extension of K/k [cf. Krasner, Duke Math. J. **6**, 120–140 (1940); these Rev. **1**, 260]. *R. Hull* (Vancouver, B. C.).

Albert, A. A. The radical of a non-associative algebra. Bull. Amer. Math. Soc. **48**, 891–897 (1942). [MF 7506]

A non-associative algebra $\mathfrak{A}$ is called semi-simple if it is a direct sum of simple algebras none of which are zero algebras. Let $\mathfrak{A}$ be a non-associative algebra which is homomorphic to a semi-simple algebra. In particular $\mathfrak{A}$ may be any algebra with a unity. Then the author demonstrates the existence of an ideal $\mathfrak{N}$, called the radical of $\mathfrak{A}$, having the following properties: (1) $\mathfrak{A}-\mathfrak{N}$ is semi-simple and (2) if $\mathfrak{B}$ is any ideal in $\mathfrak{A}$ such that $\mathfrak{A}-\mathfrak{B}$ is semi-simple, $\mathfrak{B}\geqq\mathfrak{N}$. The ideal may be obtained as follows: Let $T(\mathfrak{A})$ be the algebra generated by the transformations $x\to ax$, $x\to xa$ and the identity and let $\mathfrak{Z}$ be the radical of $T(\mathfrak{A})$. Then $\mathfrak{A}-\mathfrak{A}\mathfrak{Z}$ is a direct sum of a semi-simple algebra and a uniquely determined zero algebra. The latter has the form $\mathfrak{N}-\mathfrak{A}\mathfrak{Z}$, $\mathfrak{N}$ the radical of $\mathfrak{A}$. It may be remarked that in place of $T(\mathfrak{A})$ the author could have used in this connection the algebra generated by the mapping $x\to xa$, $x\to ax$ without the addition of the identity. The use of this algebra would have obviated the necessity of modifying the known results. [Cf. N. Jacobson, Duke Math. J. **3**, 544–548 (1937).] The author also considers isotopy of the radical and gives an example of an algebra whose radical is a field. *N. Jacobson* (Chapel Hill, N. C.).

Malcev, A. On the representation of an algebra as a direct sum of the radical and a semi-simple subalgebra. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 42–45 (1942). [MF 7627]

The author establishes a uniqueness theorem for the Wedderburn structure theorem for an associative algebra G. First define a (generalized) inner automorphism of G thus: if b,b' in G are such that $bb'=b'b=b+b'$, the correspondence $x\to x-xb-b'x-b'xb$ is an inner automorphism. In case G has a unity element e, this is the ordinary inner automorphism induced by $e-b$. Suppose then that G has radical R, and is such that the simple components of G/R have centers separable over the base field. Then any two representations $G=L_1+R$ and $G=L_2+R$ of G as a sum of R and a semisimple algebra L_1 isomorphic to G/R are conjugate under an inner automorphism induced by an element of R. [The author speaks of $G=L_1+R$ as a "direct sum," but apparently means only the usual Wedderburn decomposition.] A similar theorem for a connected Lie group G with radical R is given: Any two representations of G as direct products $G=L_1\times R=L_2\times R$, with L_1 and L_2 semisimple, are conjugate under an element of R. *S. MacLane* (Cambridge, Mass.).

NUMBER THEORY

Aucoin, Anthony A. Homogeneous and nonhomogeneous Diophantine equations. Bull. Amer. Math. Soc. **48**, 933–937 (1942). [MF 7515]

In a previous paper [Bull. Amer. Math. Soc. **45**, 330–333 (1939)], Aucoin and Parker discussed the Diophantine equation $f(x)=g(y)$, where f and g are homogeneous polynomials whose degrees are relatively prime. In the present paper the results are extended to obtaining integral solutions of the equation $f(x)=g(x)$, where f and g are homogeneous polynomials with integral coefficients of an arbitrary number of variables, each polynomial involving the same number of variables. The results depend on the degrees of the polynomials in a specified number of these variables or their reciprocals. The solutions are given in terms of polynomials (with integral coefficients) involving arbitrary parameters. *I. A. Barnett* (Cincinnati, Ohio).

Hua, Loo-keng. On the least primitive root of a prime. Bull. Amer. Math. Soc. **48**, 726–730 (1942). [MF 7272]

Let $g(p)$ denote the least positive primitive root of a prime p, and $h(p)$ the numerically least such root. It is proved that $|h(p)|<2^m p^{\frac{1}{2}}$, where m denotes the number of different prime factors of $p-1$; hence if $p\equiv 1 \pmod 4$, $|g(p)|<2^m p^{\frac{1}{2}}$; if $p\equiv 3 \pmod 4$, $|g(p)|<2^{m+1}p^{\frac{1}{2}}$. This is better than a result by Vinogradow [C. R. (Doklady) Acad. Sci. URSS. Ser. A. **1930**, 7–11 (1930)]. The proof employs an average of character sums, for example, proving that

$$(A+1)^{-1}\Big|\sum_{a=0}^{A}\sum_{n=-a}^{a}\chi(n)\Big|\leqq p^{\frac{1}{2}}-(A+1)p^{-\frac{1}{2}},$$

and then manipulates the identity

$$0=\sum_{k|p-1}(\mu(k)/\phi(k))\sum_{\chi(k)}\sum_{a=0}^{|h(p)|-1}\sum_{n=-a}^{a}\chi^{(k)}(n),$$

valid when n is not a primitive root mod p. *G. Pall.*

Hua, Loo-keng. On the least solution of Pell's equation. Bull. Amer. Math. Soc. **48**, 731–735 (1942). [MF 7273]

Let x_0, y_0 be the least positive solution of $x^2-dy^2=4$, where d is a positive integer, not a square, congruent to 0 or 1 mod 4. Let $\epsilon=(x_0+y_0d^{\frac{1}{2}})/2$. The class-number formula $(d^{\frac{1}{2}}/\log\epsilon)\cdot\sum(d|n)(1/n)$ is used to prove that $\log\epsilon<d^{\frac{1}{2}}(\frac{1}{2}\log d+1)$, whence one can easily get Schur's in-

equality $\epsilon < d^{d^3}$. The proof gives another application of the averaging of character sums, here with $\chi(n)=(d|n)$, introduced in the article reviewed above. *G. Pall.*

Carlitz, L. The reciprocal of certain types of Hurwitz series. Duke Math. J. **9**, 629–642 (1942). [MF 7338]

This paper is one of a series by the author dealing with wide generalizations of "integral power series"

$$f(u)=\sum_{m=1}^{\infty} A_m u^m/m!,$$

in which the "coefficients" A_m are integers (usually considered modulo p), the theory of which was introduced by Hurwitz [Math. Ann. **51**, 196–226 (1899); Werke, vol. 2, Basel, 1933, pp. 342–373]. In the present paper the integers A_m are replaced by polynomials $A_m(x)$ in an indeterminate x with coefficients in a Galois field $GF(p^n)$, where n and the prime p are fixed throughout. The factorial $m!$ is generalized as follows: Let $a_0, a_1, a_2, \cdots$ be, in reverse order, the digits of m when written to the base p^n, and let

$$F_k = F_k(x) = \prod_{\nu=0}^{k-1} (x^{P^k} - x^{P^\nu}), \qquad P = p^n.$$

Then the function $g(m)=g(m, x)=F_1^{a_1}F_2^{a_2}F_3^{a_3}\cdots$ replaces $m!$. The author studies the arithmetic properties of the coefficients of $u/f(u)$, where $f(u)$ is such a generalized integral power series in u, when this ratio is expanded in another such series. In case the inverse function $\lambda(u)$ $(\lambda(f(u))=u)$ has its $(k+1)$st coefficient divisible by $\prod_{\nu=1}^{k}(x^{P^\nu}-x)$, a broad and complicated generalization of the celebrated von Staudt-Clausen theorem for Bernoulli numbers is obtained. [For previous papers of this series, see Duke Math. J. **7**, 62–67 (1940); these Rev. **2**, 146; ibid. **8**, 689–700 (1941); these Rev. **3**, 147; ibid. **9**, 234–243 (1942); these Rev. **3**, 271.] *D. H. Lehmer* (Berkeley, Calif.).

Titchmarsh, E. C. On the order of $\zeta(\frac{1}{2}+it)$. Quart. J. Math., Oxford Ser. **13**, 11–17 (1942). [MF 7445]

A proof of the formula $\zeta(\frac{1}{2}+it)=O(t^{19/116}\log^{1/58} t)$ is given. This proof follows van der Corput's method and makes use of a number of the author's previous results as well as one new lemma concerning the integral $\iint e^{2\pi i f(x,y)}dxdy$. Too few details are given to make the proof easily read. Apparently the estimate for $|f''(x)|$ in §3 should read $|f''(x)|>AtRa^{-3}$, considerably complicating the following partial summations. *H. S. Zuckerman* (Seattle, Wash.).

Titchmarsh, E. C. Some problems in the analytic theory of numbers. Quart. J. Math., Oxford Ser. **13**, 129–152 (1942). [MF 7913]

The problems considered are those of determining the asymptotic behavior of sums of the type

$$\sum_{n=1}^{\infty} d(n)d_3(n+r)e^{-2n\delta}$$

as $\delta\to 0$, where $d(n)$ is the number of divisors of n and $d_3(n)$ is the number of ways in which n can be expressed as a product of three factors. Asymptotic formulas for $\sum d(n)^2e^{-2n\delta}$ and $\sum d(n)d(n+r)e^{-2n\delta}$ are already known. The author points out that these can be obtained by forming the appropriate generating function, replacing it by its dominant part on each Farey arc and integrating around a circle. Without estimating the error terms, the author applies the same procedure to (1) $\sum d_3^2(n)e^{-2n\delta}$, (2) $\sum d(n)d_3(n)e^{-2n\delta}$, (3) $\sum d_3(n)d_3(n+r)e^{-2n\delta}$, (4) $\sum d_3(n)d(n+r)e^{-2n\delta}$. He also obtains true asymptotic formulas for (1) and (2) by other methods. The results agree for (2) but do not for (1). It is interesting to note that the two results for (1) differ only by a factor 255/256. The author does not have alternative methods for (3) and (4) but is led by analogy to conjecture that his result for (4) is probably true but that (3) is more doubtful. *H. S. Zuckerman* (Seattle, Wash.).

Mordell, L. J. The product of three homogeneous linear ternary forms. J. London Math. Soc. **17**, 107–115 (1942). [MF 7318]

Let $L_r=a_rx_1+b_rx_2+c_rx_3$ $(r=1, 2, 3)$ be three homogeneous linear forms of the real variables x_1, x_2, x_3 of determinant d. Two types of forms are considered: (I) the coefficients of L_1, L_2, L_3 are all real, $d=1$; (II) the coefficients of L_1 are real and those of L_2, L_3 are conjugate complex numbers, $d=i$. The author proves that the inequality $|L_1L_2L_3|<c+\epsilon$ is solvable, for any $\epsilon>0$, in integers x_1, x_2, x_3 not all zero, where $c=1/7$ in case (I) and $c=(23)^{-\frac{1}{2}}$ in case (II); in both cases, c is the best possible value of the constant on the right-hand side, and the corresponding extreme cases for L_1, L_2, L_3 are also determined. These results were discovered earlier by H. Davenport [Proc. London Math. Soc. (2) **44**, 412–431 (1938)]. The present proof is much more direct, since solutions of the inequality are actually constructed by means of a "descente infinie," whereas Davenport's results are in the nature of existence theorems. The proof depends upon a new theorem concerning the minimum of a binary cubic form, which is still unpublished.

C. L. Siegel (Princeton, N. J.).

Salem, R. and Spencer, D. C. On sets of integers which contain no three terms in arithmetical progression. Proc. Nat. Acad. Sci. U. S. A. **28**, 561–563 (1942). [MF 7647]

Let N be an integer. Denote by $V(N)$ the maximum number of integers $\leqq N$ such that no three of them form an arithmetic progression. It has been conjectured that

$$V(N)=O\left(N^{\frac{\log 2}{\log 3}}\right).$$

The authors prove that this conjecture is false. In fact they show that

$$V(N)>N^{1-(\log\log N)^{-1}}$$

for sufficiently large N. The problem whether $V(N)=o(N)$ remains open. *P. Erdös* (Philadelphia, Pa.).

Linnik, U. V. On Erdös's theorem on the addition of numerical sequences. Rec. Math. [Mat. Sbornik] N.S. **10(52)**, 67–78 (1942). (English. Russian summary) [MF 7770]

A sequence A of positive integers is called by Khintchine an essential component if for every sequence B of density $\beta>0$ the sum $A+B$ has a density $\geqq\beta+\varphi(\beta)$. Here $\beta<1$ and $\varphi(\beta)$ depends only on β and not on any other properties of B. A basic sequence is one for which there exists a fixed integer l such that every positive integer is a sum of at most l integers of the sequence. Erdös has proved [Acta Arith. **1**, 197–200 (1936); Landau, Über einige neuere Fortschritte der additiven Zahlentheorie, Cambridge Tracts, no. 35, 1937, Satz 96] that every basic sequence is an essential component. The purpose of the present paper is to give a nontrivial example of an essential component which is not a basic sequence. The sequence is constructed as follows.

Let

$$c_0=e^{11^{10}},\ N_0=[e^{(\ln c_0)^{10/9}}],\ N_k=N_0^{2^k},$$
$$n_k=[(\ln N_k)^{1/10}],\quad (k=0,1,2,\cdots);\quad N_{-2}=N_{-1}=1.$$

The set A_k, $k=0, 1, 2, \cdots$ is defined as all integers of the form X^{n_k} in the interval (N_{k-2}, N_k). The sequence Φ_1 is defined by the condition that in the interval $(1, N_k)$ it consists of all integers which belong to at least one of the sets $A_0, A_0+A_1, \cdots, A_0+\cdots+A_{k+1}$ or the original sets $A_0, A_1, \cdots$.

Next, the sets B_k are defined in the same way as the A_k but with n_k replaced by $[n_k/2]$. The sequence Φ_2 is defined by the condition that it consists of all integers which belong to at least one of the sets B_k. The sequence $\Phi=\Phi_1+\Phi_1+\Phi_2$ will be a nonbasic essential component. To prove that it is nonbasic it is only necessary to count the number of integers in the interval $(1, N_k)$ which belong to Φ. It is shown that this number is $O(N_k^\epsilon)$ for every $\epsilon>0$. The proof that Φ is an essential component is more difficult and uses two lemmas. One is due to Vinogradoff [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] **1938**, 505–524] and concerns the exponential sum $S=\sum e(\alpha x^n)$, $\alpha=a/q+\theta/q^2$. The other is due to the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **30**, 292–294 (1941); these Rev. **2**, 349] and is an extension of the sieve method.

R. D. James (Saskatoon, Sask.).

Walfisz, Arnold. Zur additiven Zahlentheorie. IX. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **9**, 75–96 (1941). (German. Russian summary) [MF 7380]

This paper is a continuation of (VI) and (VII) of the same series [Trav. Inst. Math. Tbilissi **5**, 197–253 (1938); **8**, 69–107 (1940); these Rev. **3**, 68]. In (VI) the representation of integers by thirty-two quaternary quadratic forms was discussed; in (VII) nineteen more forms were considered. In the present paper formulas are established for the number of representations of integers by twenty-two more forms. The formulas all involve one or more of the number-theoretic functions

$$\tau(u)=\sum_{mn=u} n(3|m),\qquad 3\nmid u,$$
$$\zeta(u)=\sum_{s^2+3t^2=4u} s(-1)^{(s-1)/2},$$
$$\eta(u)=\sum_{s^2+3t^2=4u} t(-1)^{(s-1)/2}.$$

R. D. James (Saskatoon, Sask.).

Walfisz, Arnold. On lattice points in high-dimensional ellipsoids. IX. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] **10**, 111–160 (1941). (English. Russian summary) [MF 7387]

[The last communication appeared in the same Trav. **5**, 181–195 (1938).] This paper is in two parts. The first part deals with the number of lattice points in the closed ellipsoid $Q\leqq x$, where Q is a positive definite quadratic form with integral coefficients. The number of lattice points is approximated by the volume of the ellipsoid with an error denoted by $P_Q(x)$. In papers (V) and (VII) of the same series [Acta Arith. **1**, 222–283 (1936); Trav. Inst. Math. Tbilissi **5**, 1–65 (1938)] the author established for quaternary forms the existence of a positive constant $\mathfrak{S}(Q)$ satisfying

$$\int_0^x P_Q^2(w)dw=\mathfrak{S}(Q)x^3+Bx^{5/2}\log^2 x.$$

In paper (VII) $\mathfrak{S}(Q)$ was evaluated for the forms

$$(1)\qquad n_1^2+n_2^2+d(n_3^2+n_4^2),\qquad d=1,2,3,\cdots.$$

Also in (VII) the constant $\mathfrak{T}(Q)$, which appears in the formulas

$$\sum r^2_Q(n)=\mathfrak{T}(Q)x^3+Bx^2\log^2 x\quad\text{(quaternary forms)},$$
$$\sum r^2_Q(n)=\mathfrak{T}(Q)x^2+Bx^{3/2}\log x\quad\text{(ternary forms)},$$

was evaluated for the forms (1) and $n_1^2+d(n_2^2+n_3^2)$, $d=1, 2, 3, \cdots$, respectively. Here $r_Q(n)$ is the number of solutions of $n=Q$.

In the present paper the constants $\mathfrak{S}(Q)$ and $\mathfrak{T}(Q)$ are evaluated for the forms $an_1^2+bn_2^2+cn_3^2+dn_4^2$ and $\mathfrak{T}(Q)$ for the forms $an_1^2+bn_2^2+cn_3^2$. The starting point is the formula

$$\mathfrak{S}(Q)=(\pi^2/6D)\sum_{\substack{h,l=1\\(h,l)=1}}^{\infty}|T_{h,l}|^2h^{-2}l^{-6},$$

where

$$T_{h,l}=\sum_{n_i=0}^{l-1}\exp\{2\pi i(h/l)Q(n_1,\cdots,n_k)\},$$

with a similar one for $\mathfrak{T}(Q)$.

The second part of the paper deals with lattice points in a k-dimensional sphere. Using $P_k(x)$ instead of $P_Q(x)$, the author proved in paper (II) [Math. Z. **19**, 106–124 (1927)] that for $x=n!$, $n\to\infty$,

$$P_k(x)=M_kc_kx^{k/2-1}+o(x^{k/2-1}),$$

where

$$M_k=(\pi^{k/2})/(2\Gamma(k/2)),$$
$$c_k=1+i\sum_{l=2}^{\infty}\sum_{h=1}^{l-1}l^{-k}S_{h,l}^ke^{-\pi ih/l}\operatorname{cosec}(\pi h/l),$$

and $S_{h,l}$ is the well-known Gaussian sum. In this paper the series for C_k is summed for $k\equiv0, 2, 4, 6 \pmod 8$ in terms of the ζ-function and the function $L(s)$ defined by

$$L(s)=\sum_{u=1}^{\infty}(-1|u)u^{-s},\qquad s>1.$$

A typical result is

$$c_k=(L(k/2-1)-2^{1-k/2}\zeta(k/2-1))/L(k/2),\quad k\equiv2 \pmod 8.$$

Similar results are obtained for $P_k(x+r)$, $r>0$ (fixed), and $x=n!$, $n\to\infty$.

Next, inequalities are obtained which are satisfied by

$$P_k=\overline{\lim_{n\to\infty}}\,(P_k(n)/M_kn^{k/2-1}),\quad \rho_k=\lim_{n\to\infty}(P_k(n)/M_kn^{k/2-1}).$$

They are

$$P_k>1+2^{1-k/2}+2\cdot3^{-k/2},\quad \rho_k<1-2^{1-k/2}-2\cdot3^{-k/2}.$$

Similar inequalities are established under the hypotheses $n\equiv a \pmod 4$, $n\equiv b \pmod 3$, $n\equiv c \pmod 8$, $n\equiv d \pmod 5$, k even. A formula is then worked out for

$$\sum_{n=1}^{x}r_k(n)r_{k+q}(n),\quad q\geqq0;\ q\equiv0 \pmod 2),$$

analogous to the result obtained in the special case $q=0$. Finally the author points out that his work leads to two curious corollaries, one of which is

$$\sum_{n=1}^{x}r_k^2(n)\sim2\binom{k-2}{k/2-1}P_{2k}(x)$$

for $x=n!$, $n\to\infty$.

R. D. James (Saskatoon, Sask.).

Bullig, G. Zur Kettenbruchtheorie im n-Dimensionalen. (Z3). Math. Ann. 118, 1–31 (1941). [MF 6307]

The author extends the methods and results of his paper (Z1) [Abh. Math. Sem. Hansischen Univ. 13, 321–343 (1940); these Rev. 2, 253. See also the author's paper (Z2) in Mitt. Math. Ges. Hamburg 8, part 2, 164–187 (1940); these Rev. 2, 351] to n dimensions and investigates the structural properties of the graph Γ^1; the vertices of Γ^1 are the extreme n-dimensional parallelepipeds Q, and the sides the transformations Ω_ν which transform a Q into another one. For this purpose he splits $B(V)$, the boundary of the set V of all Q, into a set K of open topological spheres of $0, 1, \cdots, n-1$ dimensions without common points; then the 1-dimensional spheres in K generate the (infinity of) 1-dimensional cycles of Γ^1, and similarly the spheres of higher dimensions in K generate cycles in the higher graphs $\Gamma^2, \cdots, \Gamma^{n-1}$. One four-dimensional example is worked out in detail. *K. Mahler* (Manchester, England).

THEORY OF GROUPS

Miller, G. A. Automorphisms of the dihedral groups. Proc. Nat. Acad. Sci. U. S. A. 28, 368–371 (1942). [MF 7184]

The author states that the dihedral group of order 6 is the only dihedral group which admits no outer automorphisms, and that this group and the octic group are the only two groups which are their own groups of automorphisms. He claims to disprove a statement of G. Birkhoff and MacLane [A Survey of Modern Algebra, Macmillan Company, New York, 1941, in particular, p. 152; these Rev. 3, 99] that the octic group has no outer automorphisms. [redacted]

J. S. Frame (Meadville, Pa.).

Miller, G. A. The permutation groups of a general degree. Proc. Nat. Acad. Sci. U. S. A. 28, 407–410 (1942). [MF 7285]

A category of permutation groups of degree n is considered, all of whose transitive constituents are of degree 2. Such groups are Abelian of order 2^m. Moreover, if all the permutations of the group except identity are of the same degree, then this degree is $k\,2^m$, and the degree of the group is $2k(2^m-1)$. There is then one and only one such group for each choice of the positive integers k and m. *J. S. Frame.*

Beaumont, Ross A. Projections of the prime-power Abelian group of order p^n and type $(m-1, 1)$. Bull. Amer. Math. Soc. 48, 866–870 (1942). [MF 7502]

In this paper a proof of the following theorem is established: There exists a projectivity of the Abelian group G of order p^m $(2<m)$ and type $(m-1, 1)$ upon the group H which is not isomorphic to G if, and only if, H is generated by elements u,v, subject to the relations: $u^{p^{m-1}}=v^p=1$, $v^{-1}uv=u^{1+p^{m-2}}$; and $3<m$ in case $p=2$. *R. Baer.*

Piccard, Sophie. Quelques propositions concernant les bases du groupe symétrique et du groupe alterné. Enseignement Math. 38, 276–286 (1942). [MF 7203]

This paper continues the investigation by the author of necessary and sufficient conditions that two permutations should generate the symmetric or alternating group [cf. Comment. Math. Helv. 12, 130–148 (1939–40); these Rev. 1, 161; Mathematica, Timişoara 17, 147–166 (1941); these Rev. 4, 1]. *G. de B. Robinson* (Ottawa, Ont.).

Rutherford, D. E. On the relations between the numbers of standard tableaux. Proc. Edinburgh Math. Soc. (2) 7, 51–54 (1942). [MF 7494]

Proofs by induction are given of some of the familiar relations connecting the degrees of the irreducible representations of the symmetric group on n symbols.

G. de B. Robinson (Ottawa, Ont.).

Tovbin, A. V. On the structure of groups containing alternative of symmetric subgroups. Rec. Math. [Mat. Sbornik] N.S. 10(52), 3–6 (1942). (Russian. English summary) [MF 7764]

Let $\mathfrak{B}$ be a subgroup of the symmetric group $\mathfrak{S}_n$ which contains the alternative group $\mathfrak{A}_k$, but does not contain $\mathfrak{A}_{k+1}$. Let M denote the set of k symbols generating the group $\mathfrak{A}_k$ contained in $\mathfrak{B}$. The author remarks that an element of $\mathfrak{B}$ either permutes the symbols of M among themselves, or else it sends each of them into a symbol not belonging to M. A similar statement holds for groups containing the symmetric group $\mathfrak{S}_k$. A consequence of these remarks is that, for $k>n/2$, any group of degree n containing $\mathfrak{A}_k(\mathfrak{S}_k)$, but not containing $\mathfrak{A}_{k+1}(\mathfrak{S}_{k+1})$, is a subgroup of $\mathfrak{S}_k\mathfrak{S}_{n-k}$. This leads to a complete enumeration of the subgroups of $\mathfrak{S}_n$ containing $\mathfrak{A}_k$ or $\mathfrak{S}_k$ $(k>n/2)$.

W. Hurewicz (Providence, R. I.).

Tovbin, A. V. A generalization of Bertrand's theorem on the theory of substitution groups. Rec. Math. [Mat. Sbornik] N.S. 10 (52), 7–10 (1942). (Russian. English summary) [MF 7765]

The following theorems concerning subgroups $\mathfrak{B}$ of the symmetric group $\mathfrak{S}_n$ are proved. (I) If the index of $\mathfrak{B}$ in $\mathfrak{S}_n$ is less than $2^{[(n-1)/2]}$ and $n>3$, then $\mathfrak{B}\supset\mathfrak{A}_{[(n-1)/2]+1}$. (II) If the index of $\mathfrak{B}$ in $\mathfrak{S}_n$ is less than $3^{[(n-1)/3]}$ and $n>4$, then $\mathfrak{B}\supset\mathfrak{A}_{2\cdot[(n-1)/3]+1}$. Combining this result with the results of the preceding paper, one can obtain a complete enumeration of all the groups $\mathfrak{B}$ whose indices in $\mathfrak{S}_n$ are less than the greatest of the numbers $2^{[(n-1)/2]}$, $3^{[(n-1)/3]}$.

W. Hurewicz (Providence, R. I.).

Levi, F. W. Groups in which the commutator operation satisfies certain algebraic conditions. J. Indian Math. Soc. (N.S.) 6, 87–97 (1942). [MF 7688]

The commutator of the elements a and b in the group G is the element $(a,b)=aba^{-1}b^{-1}$. It is shown that the associative law $((a,b),c)=(a,(b,c))$ is satisfied by the commutator operation in the group G if, and only if, the central quotient group of G in Abelian. This class of groups has been investigated extensively [cf., e.g., R. Baer, Trans. Amer. Math. Soc. 44, 357–412 (1938), where further references may be found]. A larger class of groups is characterized by the alternative law: $((a,b),c)=(a,(b,c))$ whenever two of the three elements a,b and c are equal. If, for example, every element x in the group G satisfies $x^3=1$, then this alternating law is satisfied by the commutators in G. If, conversely, the alternating law holds in G, then the second commutator group $(G,(G,G))$ is part of the central of G and its elements are of orders 1 and 3. *R. Baer* (Urbana, Ill.).

Thrall, R. M. and Nesbitt, C. J. On the modular representations of the symmetric group. Ann. of Math. (2) **43**, 656–670 (1942). [MF 7398]

The general theory of the modular representations of a finite group [cf. R. Brauer, Amer. J. Math. **64**, 401–420 (1942); these Rev. **4**, 1] is here applied to the symmetric group S_m, where $p \leqq m < 2p$. The restriction on p is a result of existing limitations of the theory; there is no problem for $m < p$. The reduction of the modular regular representation is carried through with the help of the Cartan basis system and the "elementary modules" of Scott [Ann. of Math. (2) **43**, 147–160 (1942); these Rev. **3**, 263]. Nakayama's theorem [Jap. J. Math. **17**, 165–184, 411–423 (1941); these Rev. **3**, 195, 196. The reference in the first of these reviews is taken from a reprint. Actually the paper appeared in vol. **17**, pp. 165–184] relating the modular representations of S_m to Young's tableaux makes the results more explicit. In the case $m = p$ Young's semi-normal form of an ordinary irreducible representation is used. *G. de B. Robinson.*

Thrall, R. M. On the decomposition of modular tensors. I. Ann. of Math. (2) **43**, 671–684 (1942). [MF 7399]

The author applies the results obtained in the preceding paper by Thrall and Nesbitt to the reduction of the Kronecker mth power representation π_m of the full linear group over a modular field. Extending the argument of Schur's classical dissertation, he shows that an indecomposable constituent of π_m appears with a multiplicity equal to the degree of either the corresponding ordinary or modular irreducible representation of the symmetric group S_m. As before, it is assumed that $m < 2p$.

G. de B. Robinson (Ottawa, Ont.).

Hooke, Robert. Linear p-adic groups and their Lie algebras. Ann. of Math. (2) **43**, 641–655 (1942). [MF 7397]

If K is a p-adic field, the group G_K of nonsingular p-adic matrices is a locally compact zero dimensional group relative to the naturally defined topology. The present paper develops a Lie theory for p-adic linear groups, that is, closed subgroups of G_K. As in the real case, two closed subgroups H' and H'' of G_K are considered equivalent if their intersection is open in each. Unlike the situation in the real case, in the present case it is known that any zero-dimensional locally compact group contains arbitrarily small open subgroups. For this reason the present "Lie theory" deals with whole groups rather than with local groups as in the classical case. The important result of the paper is that if K is the field of ordinary p-adic numbers then there is a (1-1) correspondence between the classes of equivalent closed subgroups and the subalgebras of the Lie algebra of p-adic matrices. The same result holds for any p-adic K provided that the groups are defined by analytic functions. These results enable one to apply the whole theory of p-adic Lie algebras to the study of p-adic groups. As an instance of this, the author applies the known theory of simple Lie algebras to obtain the p-adic linear groups that are simple in the sense that their invariant subgroups are either discrete or open. *N. Jacobson.*

Eilenberg, Samuel and MacLane, Saunders. Natural isomorphisms in group theory. Proc. Nat. Acad. Sci. U. S. A. **28**, 537–543 (1942). [MF 7643]

A vague idea of covariance and contravariance is often met with in group-theory, topology, etc.; that is, one feels that the character-group is contravariant to the group, that the homology and co-homology groups of a complex are, respectively, covariant and contravariant to the complex. This is of special importance in the building up of limits of direct and inverse systems ("projective" and "inductive" limits) of groups, spaces, etc. The authors have succeeded in finding for this a precise definition, which is likely to be helpful in classifying and systematizing known results and also in looking for new relations between groups. In this note, they give a brief sketch of their method, for groups only. The main idea is that of a functor, which will best be explained by an example: for them, the definition of the character-group to an Abelian group G is only one half of the definition of a functor, which they call $Ch\,(G)$, the other half being the (obvious) rule by which any homomorphism of G into another group H determines a homomorphism of the character-group of H into the character-group of G. Generally speaking, a functor, associated with some groups $G_1, G_2, \cdots$, consists of the definition of some associated group, together with a rule indicating that the latter behaves in a certain prescribed fashion under homomorphic transformations affecting $G_1, G_2, \cdots$. Examples are given to illustrate this concept; in particular, the authors use it to derive some interesting relations concerning Whitney's "tensor-product" of groups, and clarify the nature of the latter. *A. Weil* (Bethlehem, Pa.).

Stiefel, E. Über eine Beziehung zwischen geschlossenen Lie'schen Gruppen und diskontinuierlichen Bewegungsgruppen euklidischer Räume und ihre Anwendung auf die Aufzählung der einfachen Lie'schen Gruppen. Comment. Math. Helv. **14**, 350–380 (1942).

This may be considered as a clear and useful commentary on a section of E. Cartan's work on groups [see his monograph, La théorie des groupes finis et continus et l'analysis situs, Mémor. Sci. Math., no. 42, Gauthier-Villars, Paris, 1930, pp. 38–41; also his papers, Ann. Mat. Pura Appl. **4**, 209–256 (1926); ibid. **5**, 253–260 (1928)], although the author's reference to Cartan is so incomplete and casual that he may not have been fully aware of the connection. Let G be a compact connected Lie group, T a maximal connected Abelian subgroup of G; T is a torus-group, of dimension l equal to the rank of G. If N is the normalizer of T in G, N induces on T a finite group of automorphisms; on the universal covering of T, which is an l-dimensional vector-space, the latter group appears as a "crystallographic" group Γ, that is, a discontinuous group which contains a discrete group of translations of rank l. As shown by Cartan, Γ can be generated by symmetries, and the topological properties of its fundamental domain are closely connected with those of G. This geometrical situation is carefully investigated by the author, who further clarifies it by discussing a particular case with some detail. His intention seems to have been to prove as much as possible by purely topological methods, but, since he makes essential use of the results of Cartan's infinitesimal theory at several points, there is little substantial difference from Cartan's treatment, except that some use is made of a topological device first introduced by the reviewer [C. R. Acad. Sci. Paris **200**, 518–520 (1935)] in the same connection. It is announced, however, that purely topological proofs will be given in a forthcoming paper by H. Hopf. In a final paragraph, the author points out that the consideration of the group Γ at once leads to the arithmetical properties of Lie algebras which E. Witt and B. L. van der Waerden have used in their discussion of the classification of simple Lie

groups (this, of course, refers only to those algebras which correspond to compact groups, but it is a well-known, although still puzzling, fact that every simple Lie group has a compact real form). He also makes the interesting observation that to every "crystallographic" group Γ generated by symmetries as determined by H. S. M. Coxeter [Ann. of Math. (2) **35**, 588–621 (1934)] there corresponds, in the way indicated above, a Lie group G. By a curious misprint, e^{ix}, $\cos x$, $\sin x$ are printed throughout the paper instead of $e^{2\pi ix}$, $\cos(2\pi x)$, $\sin(2\pi x)$. *A. Weil.*

Malcev, A. Subgroups of Lie groups in the large. C. R. (Doklady) Acad. Sci. URSS (N.S.) **36**, 5–7 (1942). [MF 7624]

Let G be a Lie algebra over the field of real numbers, $\mathfrak{G}$ a corresponding Lie group given in the large. This paper promises a thorough investigation of the problem of characterizing by purely algebraic properties those subalgebras H of G for which the corresponding Lie subgroup $\mathfrak{H}$ of $\mathfrak{G}$ is topologically closed in $\mathfrak{G}$.

The problem is divided into these three: (1) to determine those subalgebras for which the corresponding subgroups are closed in every form of $\mathfrak{G}$, which are called absolutely closed, (2) subalgebras for which the corresponding subgroup is closed in a given linear representation of $\mathfrak{G}$, called linearly closed, (3) subalgebras for which the corresponding subgroups are closed in the simply connected form of $\mathfrak{G}$, which are called minimally closed. This paper gives a complete solution of the second of these problems. In connection with the first problem it is pointed out that the centralizer and the normalizer of an arbitrary subalgebra H in G are absolutely closed subalgebras of G, and that a subalgebra is absolutely closed if it happens to be maximal with respect to the property of possessing certain fixed elements, otherwise arbitrary, as commutants.

L. Zippin (Flushing, N. Y.).

Malcev, A. On the simple connectedness of invariant subgroups of Lie groups. C. R. (Doklady) Acad. Sci. URSS (N.S.) **34**, 10–13 (1942). [MF 7453]

The principal result is that every connected invariant subgroup of a simply connected Lie group is simply connected. It is shown, in the course of the proof, that every connected subgroup of a simply connected solvable group is simply connected and also that a connected Lie group is simply connected if and only if its radical and associated factor-group are simply connected. The author uses the familiar cross-product of two groups A and B with respect to a homomorphism of the group A onto the group of automorphisms of B; this is the totality of pairs (a, b), $a \epsilon A$, $b \epsilon B$, with the multiplication $(a_1,b_1)(a,b)=(a_1a, b_1{}^a b)$, where $b_1{}^a$ denotes the image of b_1 under the automorphism associated with a. This is applied to local groups as well as to groups in the large. *L. Zippin* (Flushing, N. Y.).

ANALYSIS

Zygmund, A. Two notes on inequalities. J. Math. Phys. Mass. Inst. Tech. **21**, 117–123 (1942). [MF 7251]

In the first inequality it is supposed that there is a linear family of functions $f(x)$ defined over (a, b) and a linear transformation $\phi=T(f)$ with ϕ defined over (c, d). It is shown that if there is a constant M such that

$$\int_c^d |d\phi(\zeta)| \leqq M \int_a^b |df(x)|$$

for all members of the family then

$$\int_c^d \{(d\phi_1(\zeta))^2+\cdots+(d\phi_m(\zeta))^2\}^{\frac{1}{2}} \leqq M\int_a^b \{(df_1(x))^2+\cdots+(df_m(x))^2\}^{\frac{1}{2}}$$

where $\phi_i=T(f_i)$. The second inequality is an elegant generalization of a theorem of S. Bernstein. If $P(z)$ is a rational polynomial of degree n and $p>1$ it is shown that

$$\left(\int_0^{2\pi} |P'(e^{i\theta})|^p d\theta\right)^{1/p} \leqq C_p n \left(\int_0^{2\pi} |\Re(Pe^{i\theta})|^p d\theta\right)^{1/p}.$$

The precise value of the constant C_p is obtained. Each of these inequalities is proved by use of random variables. The second also makes use of an earlier inequality of Tamarkin and Zygmund, and contains a theorem of Szegö.

A. C. Schaeffer (Stanford University, Calif.).

Iyengar, K. V. A deepening of the binomial inequality. J. Mysore Univ. Sect. B. **3**, 135–138 (1942). [MF 7798]

Let m denote the lesser and M the greater of the numbers n and nx^{n-1}, and let h, g and a be the harmonic, geometric and arithmetic means of m and M. For rational positive values of n and positive values of x $(x\neq 1)$, it is shown by algebraic methods that $(x^n-1)/(x-1)$ lies, respectively, in the intervals (m,h), (h,g), (g,a) and (a,M) when $\frac{1}{2}<n<1$, $0<n<\frac{1}{2}$, $n>2$ and $1<n<2$. An application gives an algebraic proof of the fact (due to I. Schur) that, as n increases through positive values, $(1+n^{-1})^{n+k}$ is decreasing when $k\geqq\frac{1}{2}$ and ultimately increasing when $k<\frac{1}{2}$. There is one bothersome misprint: read $x^{n/2-1}$ for $x^{n/2}-1$ on page 137, line 3. *R. P. Agnew* (Ithaca, N. Y.).

Brown, Arthur B. Effective parameters. Duke Math. J. **9**, 773–789 (1942). [MF 7928]

If the set of functions $\{f_i(x,\alpha)\}$ in the variables $x=(x_1, \cdots, x_n)$ and the parameters $\alpha=(\alpha_1, \cdots, \alpha_m)$ can not be represented in the form

$$f_i(x,\alpha) \underset{x,\alpha}{\equiv} F_i(x,A(\alpha)), \quad A(\alpha)=(A_1(\alpha), \cdots, A_s(\alpha)),$$

with s actually less than m, then the m parameters α_i are essential. This paper presents an algorithm for finding the number s of essential parameters. In the classical theory the F_i and the A_i are in general unknown; here, however, in the neighborhood of a given non-singular point (x^0, α^0), the author gets the representation

$$f_i(x, \alpha) \underset{x,\alpha}{\equiv} f_i(x, \beta_1, \cdots, \beta_s, \overset{0}{\alpha}_{s+1}, \cdots, \overset{0}{\alpha}_m)$$

where the β_j are functions of s of the α_i. A discussion of singular points is also included. *F. G. Dressel.*

Beckenbach, E. F. Vector analogues of Morera's theorem. Bull. Amer. Math. Soc. **48**, 937–941 (1942). [MF 7516]

A three dimensional vector function $\mathbf{X}(x)=\mathbf{X}(x_1, x_2, x_3)$ with continuous first partial derivatives in a domain D is called Newtonian provided it is both irrotational and solenoidal, that is, provided its curl and divergence both vanish identically in D. By use of Stokes' and Green's theorems, the conditions that curl $\mathbf{X}$ and div $\mathbf{X}$ vanish can

be replaced by the conditions that (i) $\int_C \mathbf{X}\cdot d\mathbf{R}=0$ for each reducible closed curve in D and (ii) $\int_S \mathbf{n}\cdot\mathbf{X}d\sigma=0$ for each reducible closed surface S in D. If $\mathbf{X}$ satisfies (i) and (ii) and is merely continuous in D, then $\mathbf{X}$ is still a Newtonian vector [for a proof of this, see H. B. Phillips, Vector Analysis, New York, 1933, pp. 177, 178]. In the present paper the author still assumes only continuity of $\mathbf{X}$ in D and, by forming averages of $\mathbf{X}$ over spheres of radius ρ and taking the limit as $\rho\rightarrow 0$, he shows that conditions (i) and (ii) can be replaced by weaker conditions, namely, (α) $\int_{C(x,r)}\mathbf{X}\cdot d\mathbf{R}=o(r^2)$ and (β) $\int_{S(x,r)}\mathbf{n}\cdot\mathbf{X}d\sigma=o(r^3)$, each holding uniformly in each sphere in D. In (β), $S(x, r)$ is the sphere of center (x) and radius r and, in (α), $\mathbf{C}(x, r)$ is the vector with components $C_j(x, r)$, where $C_j(x, r)$ is a circle of center (x) and radius r in the plane perpendicular to the x_j-axis.

W. T. Martin (Cambridge, Mass.).

Theory of Functions of Complex Variables

Tolstoff, G. Sur les fonctions bornées vérifiant les conditions de Cauchy-Riemann. Rec. Math. [Mat. Sbornik] N.S. 10(52), 79–85 (1942). (French. Russian summary) [MF 7771]

This note proves that, if $f(z)=u(x, y)+iv(x, y)$ is bounded on a domain D, and if u and v have finite partial derivatives everywhere which satisfy the Cauchy-Riemann relations almost everywhere, then $f(z)$ is holomorphic in D. In addition to the results of Menchoff [Saks, Theorie de l'integrale, Warsaw, 1933, p. 243] the author uses the following two lemmas: If $w(x, y)$ is continuous in each variable separately on a square K then the set E for which

$$\left|\frac{w(x+h, y)-w(x, y)}{h}\right|\leqq M$$

when $|h|\leqq\delta$ is closed. If P and Q possess finite partial derivatives interior to K and if almost everywhere $\partial P/\partial y=\partial Q/\partial x$, then $\int_C Pdx+Qdy=0$ where C is the boundary of any rectangle interior to K with sides parallel to the coordinate axes. For further information on the problem of this note see results by P. T. Maker [Trans. Amer. Math. Soc. 45, 265–275 (1939)]. *R. L. Jeffery.*

Maker, Philip T. The Cauchy theorem for functions on closed sets. Bull. Amer. Math. Soc. 48, 912–916 (1942). [MF 7511]

The author extends the theorem of Cauchy to functions $f(z)$ of a complex variable, defined and continuous on a bounded plane closed set E. Conditions on f are found under which for certain coverings $C_n=\sum R_{mn}$ (R_{mn} a rectangle of boundary r_{mn}) of E and for a suitably defined continuous extension f^* of f one has $\lim\int_{c_n}f^*dz=0$ (c_n is boundary of C_n). A covering C of E is a set containing E and consisting of a set of closed nonoverlapping rectangles. All rectangles mentioned are supposed to have their sides parallel to the axes. The background of this investigation is based on the notion of a "general monogenic function," due to the reviewer, and on the measure theory methods of Menchoff.

Let R be a rectangle of boundary r containing a point of E on each side. It is shown that, if $|f(z+h)-f(z)|<B|h|$ (z, $z+h$ in $F=ER$) and the Cauchy-Riemann equations hold a.e. (almost everywhere) in F, then

$$\left|\int_r f^*dz\right|<400B \text{ meas } (S-F),$$

where S is a square of least area containing R. Also, if except for a denumerable set of points

$$\limsup |f(z+h)-f(z)|\,|h|^{-1}<\infty$$

and the Cauchy-Riemann equations hold a.e., then there is a sequence $\{C_n\}$ of C-coverings for which

$$\lim_n \sum_m \left|\int_{r_{mn}} f^*dz\right|=0.$$

Corollary: if $f(z)$ has a derivative at each point except at most a denumerable set, there is a sequence of coverings C_n, tending to E, for which $\int_{c_n}f^*dz$ tends to zero.

W. J. Trjitzinsky (Urbana, Ill.).

Boas, R. P., Jr. Entire functions of exponential type. Bull. Amer. Math. Soc. 48, 839–849 (1942). [MF 7498]

This paper formed the basis of an address delivered before the American Mathematical Society. An entire function $f(z)$ is said to be of exponential type if, for some $c\geqq 0$, and every $\epsilon>0$, there is a number $A(\epsilon)$ such that for all z

$$|f(z)|<A(\epsilon)e^{(c+\epsilon)|z|}. \tag{1}$$

The smallest c which can be used is called the type of $f(z)$. The author discusses a selection of the properties of $f(z)$ imposed by the restriction (1). Many of these results the author established, and for the most part published elsewhere, during the past five years. An excellent list of references to other authors on this subject is included.

Many of the results mentioned have to do with relations between the behavior of $f(z)$ and that of functions obtained by operating on $f(z)$ with a sequence of operators L_n, distributive and continuous, which transform an entire function of exponential type c into another function of type less than c, and which are permutable with differentiation. Associated with L there is a function $\lambda(w)$, analytic in $|w|<c$, such that, if $f(z)$ has Pólya's representation

$$f(z)=\int_C e^{zw}\phi(w)dw, \tag{2}$$

where C is any contour containing $|w|=c$ in its interior, then

$$L[f(z)]=\int_C e^{zw}\phi(w)\lambda(w)dw. \tag{3}$$

The author discusses a uniqueness problem: given a sequence of complex numbers $\{a_n\}$, for what class of functions of exponential type does $L_n[f(a_n)]=0$ ($n=0, 1, 2, \cdots$) imply $f(z)\equiv 0$? Again, given a set of circles C_n, is at least one of the functions $L_n[f(z)]$ necessarily univalent in the corresponding C_n? The theorems reduce to theorems of closure for sets of analytic functions. If L_t is a one-parameter family of operators, given sequences $\{a_n\}$ and $\{\lambda_n\}$, any function $f(z)$ of sufficiently small type, such that $L_{\lambda_n}[f(a_n)]=0$, vanishes identically. If the elements $L_{\lambda_n}[f(a_n)]$ are restricted in growth in some way, and if M_t is another family of operators, a problem presents itself as to what can be said about the growth of a sequence $\{M_{\mu_n}[f(b_n)]\}$. Special examples which have been studied are cited. Theorems involving growth properties of $M[f(x)]$ following from those of $L[f(\lambda_n)]$ are discussed. For example, if M and L are the identity operators, $|\lambda_n-n|<A$, $|\lambda_m-\lambda_n|\geqq\delta$ ($m\neq n$; $n=0, 1, 2, \cdots$), then $f(z)$, of type less than π in $x>0$ and bounded at $\{\lambda_n\}$, is bounded on the real axis. If $M[f(x)]=f'(x)$, the analogous result is that, if $|f(n)|\leqq k$, and f is of type $c'<\pi$, then $|f'(x)|\leqq c'A(c')k$, where $A(c')$ is a number, independent of f, such that

$$|M[f(x)]|\leqq A(c')\sup_{-\infty<x<\infty}|f(x)|.$$

Some illustrations of restrictions imposed on the oscillation of $f(x)$ on a line by its integrability properties are discussed, including results of the author, of Plancherel and of Pólya. *M. S. Robertson* (New Brunswick, N. J.).

Boas, R. P., Jr. and Schaeffer, A. C. A theorem of Cartwright. Duke Math. J. 9, 879–883 (1942). [MF 7937]

A theorem of M. L. Cartwright states that if $f(z)$ be an entire function of exponential type, for which

$$f(z)=O(e^{k|z|}) \quad (|z|\to\infty;\ 0<k<\pi),$$
$$|f(n)|\leqq M \quad (n=0,\ \pm1,\ \pm2,\ \cdots),$$

then $f(z)$ is bounded on the real axis, $|f(x)|\leqq A(k)M$ $(-\infty<x<\infty)$. The authors show that the order of $A(k)$ is precisely $\log\{1/(\pi-k)\}$ as k approaches π. More precisely, if $A(k)$ is the least possible constant for which Cartwright's theorem is true, then

$$\frac{2}{\pi}\log\left\{\frac{4}{\pi}\left[\frac{\pi}{\pi-k}\right]\right\}\leqq A(k)\leqq 4+2e\log\frac{\pi}{\pi-k}.$$

M. S. Robertson (New Brunswick, N. J.).

Shah, S. M. The lower order of the zeros of an integral function. J. Indian Math. Soc. (N.S.) 6, 63–68 (1942). [MF 7684]

Shah, S. M. The maximum term of an entire series. Math. Student 10, 80–82 (1942). [MF 7956]

The lower order λ of an entire function $f(z)$ of finite order is

$$\lambda=\varliminf_{r\to\infty}\frac{\log\log M(r)}{\log r}.$$

Similarly the author defines

$$\lambda_1=\varliminf_{r\to\infty}\frac{\log^+ n(r)}{\log r}.$$

In the first paper the author shows that if $f(z)$ is of order ρ, $0<\rho<1$, then $\lambda_1\leqq\lambda\leqq\lambda_1/(1+\lambda_1-\rho)$; he shows by examples that it is possible to have $\lambda_1<\lambda$ or $\lambda=\lambda_1/(1+\lambda_1-\rho)$. If $\{r_n\}$ is the sequence of absolute values of the zeros of $f(z)$, arranged in increasing order, then

$$\lambda_1\leqq\rho_1\varliminf_{n\to\infty}\frac{\log r_n}{\log r_{n+1}},$$

where ρ_1 is related to λ_1 as ρ is to λ. A factorization of interest in connection with this result is given for functions of irregular growth $(\lambda<\rho)$.

In the second paper it is shown that

$$\varliminf_{r\to\infty}\frac{\nu(r)}{\log\mu(r)}\leqq\lambda,$$

where $\mu(r)$ is the maximum term in the power series for $f(z)$ and $\nu(r)$ is its index. The corresponding result with λ replaced by ρ was known. Some relations are given involving the upper and lower limits of $r^{-\rho}$ multiplied by $n(r)$, by $\nu(r)$, and by $\log M(r)$. *E. S. Pondiczery* (Princeton, N. J.).

Ganapathy Iyer, V. A property of the maximum modulus of integral functions. J. Indian Math. Soc. (N.S.) 6, 69–80 (1942). [MF 7685]

Let $f(z)$ be an integral function and $M(r,f)=\max|f(z)|$ for $|z|\leqq r$. Then ρ, the order of $f(z)$, is defined as

$$(*)\qquad \limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r}.$$

Here we discuss some of the author's results when $0<\rho<\infty$, omitting his results for $\rho=0$. Let E be a measurable set in $(0,\infty)$, and let $E(r)$ be the measure of the portion of E in $(0,r)$. The lim sup and lim inf of $E(r)/r$ as $r\to\infty$ are called, respectively, the upper and lower density of E. A set E is called a set of regularity for the function $f(z)$ if

$$\lim\frac{\log\log M(r,f)}{\log r}=\rho$$

as $r\to\infty$ over E. The author then proves that there exists a set of regularity of $f(z)$ whose upper density is one. The author also proves that if a function $f(z)$ has a set of regularity of positive lower density then the lim sup in $(*)$ can be replaced by lim. The proofs are elementary and brief. The author gives other similar theorems. *N. Levinson.*

Fry, Cleota G. and Hughes, H. K. Asymptotic developments of certain integral functions. Duke Math. J. 9, 791–802 (1942). [MF 7929]

Asymptotic developments, in sectors around $z=\infty$, are obtained for

$$(1)\qquad E_\alpha(z;\theta,\beta)=\sum_{n=0}^{\infty}\frac{z^n}{(n+\theta)^\beta\Gamma(\alpha n+1)}$$

and for functions of type

$$(2)\qquad F_\alpha(z)=\sum_{n=0}^{\infty}\frac{h(n)z^n}{\Gamma(\alpha n+t)},$$

where the analytic function $h(z)[=h(n)$ for $z=n]$ satisfies suitable conditions. These extend earlier results of Mittag-Leffler, Barnes, W. B. Ford and Wright. Many of the results are common with those of Wright [Philos. Trans. Roy. Soc. London, Ser. A. 238, 423–451 (1940); these Rev. 1, 212; Quart. J. Math., Oxford Ser. 11, 36–48 (1940); these Rev. 2, 285; Proc. London Math. Soc. (2) 46, 389–408 (1940); these Rev. 2, 286], but the methods used are different. Proofs are based on an extension of some general theorems of W. B. Ford [The Asymptotic Developments of Functions Defined by Maclaurin Series, University of Michigan Studies, Scientific Ser., vol. 11, Ann Arbor, 1936] and Newsom [Amer. J. Math. 60, 561–572 (1938)] on asymptotic developments of functions $f(z)=\sum_0^\infty g(n)z^n$, with $g(z)$ $[=g(n)$ for $z=n]$ suitably restricted. The asymptotic expansions obtained by the authors are too lengthy to reproduce here. *I. M. Sheffer* (State College, Pa.).

Spencer, D. C. A function-theoretic identity. Amer. J. Math. 65, 147–160 (1943). [MF 7782]

If $g(R)$ is absolutely continuous for $R\geqq0$ and if

$$G(R)=\int g(R)d(\ln R),$$

then

$$r(d/dr)\int_{-\pi}^{\pi}G(|f(re^{i\phi})|)d\phi=\int_0^r\int_{-\pi}^{\pi}g'(|f|)\cdot|f|^{-1}\cdot|f'|^2\rho d\rho d\phi$$
$$+2\pi g(0)n(r,0)-2\pi g(\infty)n(r,\infty)$$

for functions f meromorphic in $|z|\leqq r$ and where $n(r,w)$ denotes the number of roots of $f=w$ in $|z|<r$. This general result of the author contains Jensen's formula when $g(R)=1$, the identity of Hardy-Stein for $g(R)=\lambda R^\lambda$, $\lambda>0$. Ahlfors' formulation of Nevanlinna's first main theorem

may also be derived from the result. Define

$$p(r, R)=(2\pi)^{-1}\int_{-\pi}^{\pi} n(r, Re^{i\phi})d\phi,$$

$$n(w)=n(1, w)=\lim_{r\to 1} n(r, w),$$

$$\phi(R)=p(1, R)=\lim_{r\to 1} p(r, R)=(2\pi)^{-1}\int_{-\pi}^{\pi} n(Re^{i\phi})d\phi.$$

If, for all $R_1>0$, $p>0$,

$$\int_0^{R_1} p(R)d[\psi(R)]\leqq p\int_0^{R_1} d[\psi(R)],$$

f is said to be mean p-valent (ψ). The author has developed elsewhere the special case of ordinary mean p-valency, where $\psi=R^2$. He now shows that, if f is mean p-valent (ψ), then

$$\int_0^{\infty} \phi\{\psi(R)\}d[-p(R)]\geqq p\phi\left\{p^{-1}\int_0^{\infty} \psi(R)d[-p(R)]\right\}$$

if $\phi''\geqq 0$, $\phi(0)\leqq 0$. The inequality becomes $\leqq$ when $\phi''\leqq 0$, $\phi(0)\geqq 0$. If f is mean p-valent and if $g([R]^{\frac{1}{2}})$ is convex, $g(0)\leqq 0$, then

$$r(d/dr)(2\pi)^{-1}\int_{-\pi}^{\pi} G(|f|)d\phi\geqq pg\left\{\left[p^{-1}\int_0^{\infty} p(r, R)d(R^2)\right]^{\frac{1}{2}}\right\}.$$

The inequality is reversed when $g(0)\geqq 0$, $g([R]^{\frac{1}{2}})$ concave.

For $0<\alpha<\pi/2$ let $\Gamma_\alpha(\theta)$ denote the domain $|z|<1$, $|e^{i\theta}-z|<5/4$, $|\arg(e^{i\theta}-z)-\theta|<\alpha$. Let

$$S_\alpha(\theta)=\frac{1}{\pi}\iint_{\Gamma_\alpha(\theta)} \frac{|f'|^2}{(1+|f|^2)^2} r\,dr\,d\phi.$$

Then the author obtains the following localization of a well-known theorem of Nevanlinna: Suppose $f(z)$ is regular for $|z|<1$, and that to each θ of a set $\mathfrak{E}$, $0<m\mathfrak{E}\leqq 2\pi$, there is a positive number $\alpha=\alpha(\theta)$ for which $S_\alpha(\theta)$ is finite. Then for almost all θ of $\mathfrak{E}$ and for every β, $0<\beta<\pi/2$, the function $f(z)$ tends to a finite limit as $z\to e^{i\theta}$ in $\Gamma_\beta(\theta)$.

M. S. Robertson (New Brunswick, N. J.).

Spencer, D. C. Note on mean one-valent functions. J. Math. Phys. Mass. Inst. Tech. **21**, 178–188 (1942). [MF 7760]

Let d denote the upper bound of $|f(z)/z|$ for $|z|<1$ when

$$f(z)=a_1z+a_2z^2+\cdots+a_nz^n+\cdots$$

is regular for $|z|<1$. The author has shown elsewhere [Ann. of Math. (2) **42**, 614–633 (1941); these Rev. **3**, 78] for strongly mean one-valent functions $f(z)$ that $d\geqq|a_1|/4$. For the wider class of mean one-valent functions the corresponding inequality $d\geqq|a_1|/A$ probably has as the lower bound for A the number 4. In this note the author shows that $A<7$, and supplies additional data to indicate that a much better upper bound for A exists.

M. S. Robertson (New Brunswick, N. J.).

Alenitzyn, G. On the coefficients of p-valent functions. Rec. Math. [Mat. Sbornik] N.S. **10(52)**, 51–58 (1942). (Russian. English summary) [MF 7768]

The author considers the following two subclasses of functions $f(z)$ regular and p-valent in $|z|<1$: (a) the class $S_p^{(k)}$ of functions of the form

$$f(z)+z^q+a_{k+q}z^{k+q}+a_{2k+q}z^{2k+q}+\cdots,$$

where q and k are positive integers and (necessarily) $q\leqq p$; (b) the class $\bar{S}_p^{(k)}(\bar{S}_p^{(k)}\subset S_p^{(k)})$ of functions of $S_p^{(k)}$ which are nonvanishing for $0<|z|<1$. He proves the following three results. (1) If $f\epsilon S_p^{(k)}$, then $F(z)=[f(z^{1/k})]^k\epsilon S_p^{(1)}$; if $F\epsilon\bar{S}_p^{(1)}$ and $(q, k)=1$, then $f(z)=[F(z^k)]^{1/k}\epsilon\bar{S}_p^{(k)}$. (2) If $f\epsilon S_p^{(1)}$ and $|f(re^{i\theta})|=O(1-r)^{\alpha}$, then $|a_n|=O(n^{\alpha-1})$, provided that $\alpha>\frac{1}{2}$. (3) If $f\epsilon\bar{S}_p^{(k)}$, then $|a_n|=O(n^{2p/k-1})$, provided that $2p/k>\frac{1}{2}$. Results (2) and (3) (without the restriction that f is nonvanishing in $0<|z|<1$) are particular cases of more general results of the reviewer [Proc. London Math. Soc. (2) **47**, 201–211 (1941); these Rev. **3**, 79; Trans. Amer. Math. Soc. **48**, 418–435 (1940); these Rev. **2**, 82].

D. C. Spencer (Stanford University, Calif.).

Dufresnoy, Jacques. Sur les cercles de remplissage des fonctions méromorphes. C. R. Acad. Sci. Paris **214**, 467–469 (1942). [MF 7862]

The following results concerning the "cercles de remplissage" of a meromorphic function are announced: (I) If $w=f(z)$ is meromorphic in the finite plane and

$$|z|\,|f'(z)|/(1+|f(z)|^2)$$

is not bounded, then there exists a family of "cercles de remplissage" Γ_k in the sense of Valiron-Milloux. (II) In Γ_k, the equation $f(z)-a=0$ has at least a simple root for all values of a save for certain exceptional values which can be enclosed in four circles whose radii tend to zero with $1/k$. (III) Given three domains D_i, the Riemann surface corresponding to Γ_k has at least one disk on one of the D_i provided k is sufficiently large. (IV) Given $q(\geqq 3)$ simply-connected, mutually disjoint domains D_i, with each one of which is associated a positive whole number μ_i such that $\sum[1-1/\mu_i]>2$, then for k sufficiently large the Riemann surface corresponding to Γ_k has at least one disk of less than μ_i sheets on one of the domains D_i. Related results subject to more restrictive hypotheses are discussed.

M. H. Heins (Chicago, Ill.).

Lelong, Pierre. Sur certaines fonctions multiformes. C. R. Acad. Sci. Paris **214**, 53–54 (1942). [MF 7834]

Let $y(x)$ be a multiform function in the complex variable x defined by the relation $f(x, y)=0$, where $f(x, y)$ is holomorphic in x, y for $|x|<\infty$ and y contained in a connected domain D. If $f(x, a)$ does not vanish in the x-plane, a is called an exceptional value of $y(x)$. The author obtains information on the set E of these exceptional values. If G is a subset of D, let $M(r, G)$ denote the maximum of $|f(x, y)|$ for $|x|<r$, y in G. If, for any closed domain G interior to D, $M(r, G)$ is of finite order, $f(x, y)$ is said to be of finite order in x in D. The author announces in this note that in this case the set E is denumerable and has no limit point in D. When D is the entire plane this result may be made precise. Now $M(r, G)$ is of order μ for all G. Let q be the integer immediately above μ. Define

$$\psi_q(x, y)=D_x^q[\log f(x, y)].$$

Then the points of E are amongst the zeros of $\psi_q(0, y)$.

M. S. Robertson (New Brunswick, N. J.).

Ferrand, Jacqueline. Sur les fonctions holomorphes ou méromorphes dans une couronne. C. R. Acad. Sci. Paris **214**, 50–52 (1942). [MF 7830]

Employing the results of H. Cartan and Ahlfors [Ahlfors, Soc. Sci. Fenn. Comment. Phys.-Math. **5**, no. 16, 1–19 (1931)] concerning additive set functions, the author obtains the following generalization of results due to Denjoy [C. R. Acad. Sci. Paris **212**, 1071 (1941); **213**, 15, 115

(1941)], Wolff [Nederl. Akad. Wetensch., Proc. **44**, no. 8 (1941)] and Dufresnoy [C. R. Acad. Sci. Paris **213**, 393 (1941)]. Let $f(z)$ denote a function analytic for $\rho \leqq |z| < 1$, let $S(r)$ denote the area of the Riemann surface corresponding to $\rho \leqq |z| \leqq r(<1)$ and let $\phi(r)$ denote a continuous positive function such that $\int_\rho^1 \phi(r) dS(r)$ is bounded as $r \to 1$. Then, for all points a of $|z|=1$, except for a set of linear measure zero,

$$f'(z) = o[(z-a)^{\frac{1}{2}}/((1-r)(\phi(r))^{\frac{1}{2}})]$$

when $r \to 1$. If $\phi \equiv 1$, the theorems of Denjoy and Wolff are obtained without restrictions required by these authors. If $\phi \equiv (1-r)^{1-\delta}$ $(0 \leqq \delta < 2)$, results related to those of Dufresnoy were obtained. This theorem and related ones can be extended to meromorphic functions provided that $S(r)$ is replaced by the spherical area of the Riemann surface corresponding to $\rho \leqq |z| \leqq r$ and $|f'(z)|$ is replaced by $|f'(z)|/(1+|f(z)|^2)$. *M. H. Heins* (Chicago, Ill.).

Ferrand, Jacqueline. Sur la représentation conforme. C. R. Acad. Sci. Paris **214**, 250–253 (1942). [MF 7846]

Continuing the investigations of the preceding note, the author considers the following situation. The function $f(z)$ is assumed analytic in $z = x+iy$ for $0 < \Re z \leqq x_0$; $\phi(x)$ is assumed positive and continuous. It is supposed that f is such that

$$\int_0^{x_0} \int_{-\infty}^{+\infty} \varphi(x) |f'(x+iy)|^2 dx dy < +\infty.$$

The following notation is employed: $a = it$ is a point of the imaginary axis; $c_{\rho,a}$ is the semi-circumference $|z-a| = \rho$, $|\arg (z-a)| < \pi/2$; $\lambda(\rho, a)$ is the length of $\gamma_{\rho,a}$ described by $f(z)$ when z describes $c_{\rho,a}$.

If $\int_{-\pi/2}^{+\pi/2} \rho d\theta/\phi(\rho \cos \theta) = g(\rho) < +\infty$, it is shown that, at every a, $\int_0^{\rho} (\lambda^2(\rho, a)/g(\rho)) d\rho$ is bounded. From this result it is concluded that $\lambda(\rho, a)$ is bounded if $0 < \rho \leqq x_0$, except for a set e of ρ values on which $\int_e d\rho/g(\rho)$ is arbitrarily small. Further, if $\int_0^{x_0} dx/\phi(x) < +\infty$, it is deduced that, except for a set of points of measure zero, the length of the curve described by $f(z)$ when z describes the segment $(y=t, x_0 = x > 0)$ is bounded. The function $f(z)$ is then specialized to be univalent and various results related to the Carathéodory theory of prime-ends are obtained. *M. H. Heins.*

Subba Rao, M. V. Some elliptic function formulae. Math. Student **10**, 87–90 (1942). [MF 7959]

Starting with well-known formulae for $\wp'(u+\omega_k)$, $(k=1, 2, 3)$, the author derives, in a simple manner, the following formulae:

$$\frac{1}{\rho'}\left(\frac{\rho''}{\rho'}\right)^s + \frac{1}{\rho_1'}\left(\frac{\rho_1''}{\rho_1'}\right)^s + \frac{1}{\rho_2'}\left(\frac{\rho_2''}{\rho_2'}\right)^s + \frac{1}{\rho_3'}\left(\frac{\rho_3''}{\rho_3'}\right)^s \equiv \phi(u),$$

where

$$\phi(u) \equiv \begin{cases} 0 & \text{for } s=0, 1, 2, 4 \\ 32 & \text{for } s=3 \\ 768\, \wp'(2u) & \text{for } s=5, \end{cases}$$

and

$$\rho = \wp(u), \quad \rho_k = \wp(u+\omega_k).$$

As a consequence of the formulae corresponding to $s=0, 1, 2, 4$ the following is obtained:

$$\begin{vmatrix} a, & a^2, & a^4, & 1 \\ b, & b^2, & b^4, & 1 \\ c, & c^2, & c^4, & 1 \\ d, & d^2, & d^4, & 1 \end{vmatrix} = 0,$$

where $a = \rho''/\rho'$, $b = \rho_1''/\rho_1'$, $c = \rho_2''/\rho_2$, $d = \rho_3''/\rho_3$.

M. A. Basoco (Lincoln, Neb.).

Fueter, Rudolf. Über einen Hartogs'schen Satz in der Theorie der analytischen Funktionen von n komplexen Variablen. Comment. Math. Helv. **14**, 394–400 (1942).

In this paper the author extends his theory of right and left regular analytic functions of quaternion variables to that of n complex variables and, as in his previous paper [Comment. Math. Helv. **12**, 75–80 (1939); these Rev. **1**, 115], applies his results to the well-known theorem of Hartogs for n complex variables. The main theorem proved is: If $w = f(z)$, z representing n hypercomplex variables, is a right regular analytic function at each point of a closed bounded hypersurface R possessing direction, which does not cut itself anywhere, then w is right regular everywhere in the interior of R. The proof of this theorem is merely outlined, the details of which will be published in a dissertation at the University of Zurich. *A. Gelbart.*

Bers, Lipman. On bounded analytic functions of two complex variables in certain domains with distinguished boundary surface. Amer. J. Math. **64**, 514–530 (1942). [MF 6946]

In this paper the author obtains certain generalizations to functions of two complex variables of theorems of the Fatou type for a class of domains ($\mathfrak{M}$), possessing a distinguished boundary surface ($\mathfrak{F}$), of the form $z_1 \epsilon \mathfrak{B}(z_2)$, $|z_2| < 1$, where $\mathfrak{B}(z_2)$ is the interior of a simple closed curve $C(z_2)$: $z_1 = h(z_2, \lambda)$, $0 \leqq \lambda < 2\pi$; $h(z_2, \lambda)$ is analytic in z_2 for each λ [see Bergman, Math. Ann. **104**, 611–636 (1931)]. Though the bicylinder belongs to this class, domains of this class cannot in general be obtained by a pseudo-conformal transformation of the bicylinder.

Let $G(z_2; z_1, z_1')$ be the Green's function of the domain $\mathfrak{B}(z_2)$ and $n(z_2, \lambda)$ the direction of the inner normal to the curve $C(z_2)$ at the point $h(z_2, \lambda)$. Set

$$P(se^{it}, \varphi) = \frac{1}{2\pi} \frac{1-s^2}{1-2s\cos(t-\varphi)+s^2},$$

$$\tilde{P}(z_2; z_1, \lambda) = \frac{1}{2\pi} \frac{\partial G[z_2; z_1, h(z_2, \lambda)]}{\partial n(z_2, \lambda)} \frac{\partial h(z_2, \lambda)}{\partial \lambda},$$

$$Q(z_1, z_2; \theta, \lambda) = \tilde{P}(z_2; z_1, \lambda) P(z_2, \theta).$$

The function Q is analogous to the kernel of the Poisson integral [Bergman, Rec. Math. [Mat. Sbornik] N.S. **1**(**43**), 851–862 (1936)]. A function $f(z_1, z_2)$ possesses a sectorial limit at $\{h(e^{i\theta}, \lambda), e^{i\theta}\} \epsilon \mathfrak{F}$ when $f(z_1, z_2)$ approaches a limit as $\{z_1, z_2\} \to \{h(e^{i\theta}, \lambda), e^{i\theta}\}$ in an interior "sectorial" region. The author proves the theorem: If $f(z_1, z_2)$ is a bounded analytic function in $\mathfrak{M}$, then f can be represented in the form

$$f(z_1, z_2) = \int_0^{2\pi} \int_0^{2\pi} Q(z_1, z_2; \theta, \lambda) F(\theta, \lambda) d\theta d\lambda,$$

where F is a bounded measurable (complex) function and f possesses the sectorial limit $F(\theta, \lambda)$ almost everywhere on $\mathfrak{F}$. From this theorem it follows that, for certain conditions of approach on the sequence, and the condition that

$$\overline{\lim}_{\nu\to\infty} |f(z_1^\nu, z_2^\nu)| \leqq \varphi(\theta, \lambda),$$

$$\log |f(z_1, z_2)| \leqq \int_0^{2\pi} \int_0^{2\pi} Q(z_1, z_2; \theta, \lambda) \log \varphi(\theta, \lambda) d\theta d\lambda;$$

ϕ is a measurable real function. As a corollary, it is shown that, if f possesses a sectorial limit zero on a set $E \subset \mathfrak{F}$ of exterior positive measure, $f \equiv 0$.

These results are obtained after first proving similar theorems for biharmonic functions (the real or imaginary parts of functions of two complex variables), making use of the concept of functions of the "extended class" [Bergman, Compositio Math. 6, 305–335 (1939)]. *A. Gelbart.*

Bergman, Stefan. Über uneigentliche Flächenintegrale in der Theorie der analytischen Funktionen von zwei komplexen Veränderlichen. Revista Ci., Lima **43**, 675–682 (1941); **44**, 131–140, 377–394 (1942). (3 plates) [MF 6641]

In the theory of functions of a complex variable, the relation

$$(2\pi i)^{-1}\int_C f(z)dz/(z-z_0)=\tfrac{1}{2}f(z_0), \qquad z_0\subset C,$$

holds under suitable conditions on the curve C and the function $f(z)$. In the present paper the author develops an analogous theory for the case of two complex variables.

For each λ in $\lambda'\leqq\lambda\leqq\lambda''$, let $E(\lambda)$ be a domain in the complex Z-plane whose boundary $b(\lambda)$ is a simple closed rectifiable curve. Let $h^{(j)}(Z,\lambda)$, $j=1,2$, be continuous differentiable functions of Z,λ in $[Z\subset E(\lambda),\ \lambda'\leqq\lambda\leqq\lambda'']$, which for each fixed value of λ are analytic functions of Z. The 1-1 mapping $\mathbf{R}$: $z_j=h^{(j)}(Z,\lambda)$, $j=1,2$, takes the set $[Z\subset E(\lambda),\ \lambda'\leqq\lambda\leqq\lambda'']$ into a three-dimensional hypersurface I. Such a hypersurface I has a two-dimensional mantel surface G which is the image by $\mathbf{R}$ of the set $[Z\varepsilon b(\lambda),\ \lambda'\leqq\lambda\leqq\lambda'']$. The author considers his improper integrals over such mantel surfaces G. He deals not only with analytic functions of two complex variables but also with a larger class $\mathbf{K}(I)$ consisting of functions $F(Z,\lambda)$ analytic in Z for each fixed value of λ. Finally, let C be a (one-dimensional) curve lying on G, C being defined as the image by $\mathbf{R}$ of the set $[Z=Z^0(\lambda),\lambda'\leqq\lambda\leqq\lambda'']$. Under suitable hypotheses on the curve C, the author proves the result

$$\frac{1}{2\pi i}\iint\limits_{R^{-1}(G)}\frac{F(Z,\lambda)}{Z-Z^0(\lambda)}dZd\lambda=\tfrac{1}{2}\int_{\lambda'}^{\lambda''}F(Z^0(\lambda),\lambda)d\lambda,$$

for $F\subset\mathbf{K}(I)$. This enables one to treat, for example, improper integrals of analytic functions $f(z_1,z_2)$ having simple poles on a curve C but otherwise regular on G.

W. T. Martin (Cambridge, Mass.).

Valeiras, Antonio. On monogenic functions of a special class of hypercomplex variables. Publ. Circulo Mat. Inst. Nac. Profesorado Secund. no. 5, 1–56 (1939). (Spanish) [MF 7497]

The author discusses a system of hypercomplex numbers which are related to Humbert's equation

$$\frac{\partial^3 v}{\partial x^3}+\frac{\partial^3 v}{\partial y^3}+\frac{\partial^3 v}{\partial z^3}-3\frac{\partial^3 v}{\partial x\partial y\partial z}=0$$

in the same way that ordinary complex numbers are related to Laplace's equation. These numbers are of the form $w=u_0x+u_1y+u_2z$ (x,y,z real), subject to the natural addition and to a (commutative) multiplication defined in such a way that the system is isomorphic to the algebra of matrices of the form

$$\begin{pmatrix} x & y & z\\ z & x & y\\ y & z & x\end{pmatrix}.$$

Two representations analogous to the polar form of a complex number are obtained, and allow multiplication to be expressed more simply. A function defined in the system with values in the system is called monogenic in a domain if its integral along every closed path in the domain vanishes; if monogenic, it is differentiable. The components of a monogenic function satisfy Humbert's equation. The logarithm (defined as an integral) and exponential (defined as a series) are discussed. Their properties are analogous to those of the complex logarithm and exponential; for example, the hypercomplex exponential function has the period $2\pi 3^{-\frac{1}{2}}(u_2-u_1)$. Partial analogues of Cauchy's integral formula are obtained. Extensions to systems with four components are mentioned briefly. *E. S. Pondiczery.*

Special Functions

Bouzitat, Jean. Sur une classe d'équations fonctionnelles. C. R. Acad. Sci. Paris **214**, 608–610 (1942). [MF 7896]

The author proves that the functional equation

$$(1)\qquad P_1[G(z)]=P_2\{G[Q(z)]\},$$

where $Q(z)$ is a polynomial of degree higher than the first, P_2 any polynomial and P_1 a polynomial of second degree, cannot be satisfied by a nonrational entire function $G(z)$. The proof reduces equation (1) to certain functional equations, for which the nonexistence of nonrational entire solutions had been shown by G. Julia [Ann. École Norm. (3) **40**, 97–150 (1923)] and G. Valiron [C. R. Acad. Sci. Paris **190**, 1113–1115, 1225–1227 (1930)]. *F. John.*

Erdélyi, A. On certain expansions of the solutions of the general Lamé equation. Proc. Cambridge Philos. Soc. **38**, 364–367 (1942). [MF 7803]

The Lamé equation is considered in the form

$$(1)\qquad d^2y/dx^2+[h-n(n+1)k^2\,\mathrm{sn}^2\,(x,k)]y=0,$$

where n and h are arbitrary parameters. Writing

$$e^{2i\omega}=\frac{\mathrm{cn}\,x\,\mathrm{cn}\,p-k'\,\mathrm{sn}\,x\,\mathrm{sn}\,p}{\mathrm{cn}\,x\,\mathrm{cn}\,p+k'\,\mathrm{sn}\,x\,\mathrm{sn}\,p},\quad z=\frac{\mathrm{dn}\,x\,\mathrm{dn}\,p}{k'},$$

and $\phi_m(x)=e^{im\omega}P_n{}^m(z)/P_n{}^m(0)$, the author shows that a solution of (1) may be written

$$y(x)=\sum_{-\infty}^{\infty}c_r\phi_{2r-\theta}(x),$$

where θ is the characteristic exponent and the constants c_r are determined by recurrence relations. Simpler forms of the series are obtained by assigning particular values to the arbitrary constant p. *M. C. Gray* (New York, N. Y.).

Erdélyi, A. Integral equations for Heun functions. Quart. J. Math., Oxford Ser. **13**, 107–112 (1942). [MF 7634]

Let $F(h,x)$ be a solution of

$$\text{(i)}\quad x(x-1)(x-a)u''+\{\gamma(x-1)(x-a)+\delta x(x-a)$$
$$+\epsilon x(x-1)\}u'+\alpha\beta(x-hu)=0,\quad 1+\alpha+\beta-\gamma-\delta-\epsilon=0,$$

which is regular in the neighborhhod of $x=0$ and equals 1

at $x=0$. The author constructs first kernels $k(x, y)$ and contours C such that the integral

$$\text{(ii)}\quad I(F)=\int_C y^{\gamma-1}(1-y)^{\delta-1}(1-y/a)^{\epsilon-1}k(x,y)F(h,y)dy$$

is a solution of (i) which is regular in the neighborhood of $x=0$. The integral equation (ii) $u=\lambda I(u)$, of which $u=F(h, x)$ is then an eigenfunction, is, in general, singular. Under certain additional conditions (one of which is that the heretofore arbitrary accessory parameter h is a root of a certain transcendental equation), (ii) becomes a Fredholm equation whose eigenfunctions are the so-called Heun functions, that is, solutions of (i) which are regular in the neighborhood of not only $x=0$ but also $x=1$. In case $\alpha=-n$, where n is a positive integer, $n+1$ of the Heun functions become polynomials, the Heun polynomials. The author constructs finally the most general kernel $k(x, y)$ all of whose eigenfunctions are Heun polynomials. The present paper contains extensions of results obtained by C. G. Lambe and D. R. Ward [Quart. J. Math., Oxford Ser. **5**, 81–97 (1934)]. *E. Rothe* (Oskaloosa, Iowa).

Jackson, F. H. On basic double hypergeometric functions. Quart. J. Math., Oxford Ser. **13**, 69–82 (1942). [MF 7631]

Four normal functions are defined by double series of the same type as those of Appell, except that the symbol $(a)_\nu$ is defined by the equation

$$(a)_\nu=(1-q^a)(1-q^{a+1})\cdots(1-q^{a+\nu-1}).$$

The four abnormal functions differ from the four normal functions in the fact that $y^nq^{\frac{1}{2}n(n-1)}$ replaces y^n in the general term. The convergence of the series is first discussed and then the fundamental transformation analogous to the addition theorem for the hypergeometric series. Expansions of certain product functions in basic double hypergeometric functions are given, together with the inverse expansions. Corresponding expansions for the abnormal series are found and exceptional expansions are noted. A duplication formula is also given. The rest of the paper is devoted to certain partial differential equations associated with the functions. *H. Bateman* (Pasadena, Calif.).

Mitra, S. C. On a certain transformation about generalised hypergeometric series. J. Indian Math. Soc. (N.S.) **6**, 84–86 (1942). [MF 7687]

A relation between two well-poised functions of the type ${}_6F_5(-1)$ is used to obtain transformations of functions of the type ${}_3F_2(+1)$. A simple result is

$${}_3F_2(c, b+m, a-1; a, b; 1)=\frac{\Gamma(a)\Gamma(b)\Gamma(1-c)\Gamma(b+m+1-a)}{\Gamma(b+m)\Gamma(1+b-a)\Gamma(a-c)},$$

where m is a positive integer and $c+m<1$. *M. C. Gray.*

Mitra, S. C. On an extension of a theorem of Watson in generalised hypergeometric series. J. Indian Math. Soc. (N.S.) **6**, 81–83 (1942). [MF 7686]

The author obtains an expression for the function ${}_3F_2(a, b, c; \frac{1}{2}(a+b+1), 2c+2m; 1)$ in terms of two well-poised ${}_7F_6$ functions. If either m or $m-\frac{1}{2}$ is a positive integer, these series can be summed by Dougall's theorem and the ${}_3F_2$ is evaluated in closed form. When $m=0$ the formula reduces to Watson's theorem. *M. C. Gray.*

Sinha, S. A few infinite integrals. J. Indian Math. Soc. (N.S.) **6**, 103–104 (1942). [MF 7690]

Using certain infinite integrals due to W. N. Bailey [Proc. London Math. Soc. (2) **40**, 37–48 (1935)], the author deduces several results of which the following is typical:

$$\int_0^\infty t^{2\rho-\mu-\nu-1}e^{-ct^2}F_\nu(at)F_\nu(bt)\,{}_1F_1\left(\frac{\rho-\lambda}{2}+1, \rho+1; ct^2\right)dt$$
$$=\frac{\rho a^\mu b^\nu\Gamma(\frac{1}{2}\lambda-\frac{1}{2}\rho)\{\Gamma(\rho)\}^2}{2^{\mu+\nu+1}c^\rho\Gamma(\mu+1)\Gamma(\nu+1)\Gamma(\frac{1}{2}\lambda+\frac{1}{2}\rho)},$$

$a+b<0$, $\Re(\lambda+\rho)>0$, $\Re(\rho-\mu-\nu)<5/2$. *M. A. Basoco.*

Sinha, S. Some infinite integrals. Bull. Calcutta Math. Soc. **34**, 67–77 (1942). [MF 7531]

The author bases his investigations on an infinite integral given by Varma [J. Indian Math. Soc. (N.S.) **3**, 25–33 (1938)], which involves, in the integrand, the Kummer function ${}_1F_1(\alpha, \beta, z)$ and Whittaker's function $W_{k,m}(z)$. This result, in conjunction with definite integrals given by Sonine, Gegenbauer, Hankel, Weber [listed in Watson, Theory of Bessel Functions, Cambridge, England, 1922] and MacRobert [Philos. Mag. (7) **26**, 82–93 (1938)], yields a collection of infinite integrals too complicated and numerous to list here. The following is a sample of the results listed:

$$\int_0^\infty \frac{J_\nu\{a(t^2+z^2)^{\frac{1}{2}}\}J_\lambda\{a(t^2+z^2)^{\frac{1}{2}}\}}{(t^2+z^2)^{\frac{1}{2}(\lambda+\nu)}}\cdot\frac{e^{-bt^2}}{t}k_{2p+2}(bt^2)dt$$
$$=\frac{(-1)^p}{2(p+1)}\frac{J_\nu(az)J_\lambda(az)}{z^{\lambda+\nu}},$$

where $k_n(x)$ denotes Bateman's function and $\Re(\nu+\lambda+\frac{5}{2})>1$, $a<0$, $\Re(p)>-2$, p a positive integer. Other similar results involving Laguerre and Hermite polynomials are also given. *M. A. Basoco* (Lincoln, Neb.).

Mohan, B. A class of infinite integrals. II. J. Indian Math. Soc. (N.S.) **6**, 98–101 (1942). [MF 7689]

Using an integral representation for the function

$$\psi_\nu(x)=\sum_{r=0}^\infty\frac{\Gamma(\frac{1}{2}+r)x^{\nu+2r}}{2^{\nu+2r}r!\Gamma(\nu+r+1)},$$

formerly obtained by the author [Bull. Calcutta Math. Soc. **33**, 99–103 (1941); these Rev. **4**, 82], certain infinite integrals are evaluated. The following result is typical:

$$\int_0^\infty x^{-p}e^{-c^2x^2/4}F(bx)\psi_{m+p+1}(cx)dx$$
$$=\frac{\pi^{\frac{1}{2}}b^{m+p-\frac{1}{2}}e^{-b^2/2c^2}}{2^pc^{m+\frac{1}{2}}\Gamma(m+p+\frac{3}{2})}W_{-p/2-\frac{1}{4},-p/2-\frac{1}{4}}(b^2/c^2),$$

where $W_{m,k}(z)$ is Whittaker's function and $\Re(m)>-1$, $\Re(m+p)>-\frac{3}{2}$, $\Re(m+2p)>-\frac{5}{2}$, $\Re(b)>0$, $\Re(c)>0$. *M. A. Basoco* (Lincoln, Neb.).

Mohan, B. Infinite integrals involving Struve's functions. III. Bull. Calcutta Math. Soc. **34**, 55–59 (1942). [MF 7529]

The object of this note is to evaluate certain infinite integrals involving Struve's function defined by [Watson, Theory of Bessel Functions, Cambridge, England, 1922]

$$H_\nu(x)=\sum_{\gamma=0}^\infty\frac{(-1)^\gamma x^{\nu+2\gamma+1}}{2^{\nu+2\gamma+1}\Gamma(\gamma+\frac{3}{2})\Gamma(\nu+\gamma+\frac{3}{2})}.$$

It is not possible to give here a complete list of results; it

may be sufficient to indicate one of the simpler formulae given:

$$\int_0^\infty x^{-\nu}K_\nu(ax)K_\nu(bx)H_\nu(bx)dx = \frac{b^{\nu+1}\sqrt{\pi}}{2^{\nu+2}a}\Gamma(\tfrac{1}{2}-\nu)F\left(\begin{matrix}\tfrac{1}{2}, \tfrac{1}{2}-\nu \\ \tfrac{3}{2}\end{matrix}; -\frac{b^2}{4a^2}\right).$$

M. A. Basoco (Lincoln, Neb.).

Watson, G. N. An infinite integral. Proc. Cambridge Philos. Soc. 38, 323–324 (1942). [MF 7812]

A new way of evaluating integrals containing Fourier kernels is based upon the remark that the general term in the power series for sin x is the residue of

$$-\pi \operatorname{cosec}(\pi t)x^{2t-1}/\Gamma(2t)$$

at $t=n$. When $x>0$ and $3<4c<4$ there is thus an absolutely convergent integral of Mellin's type

$$2i \sin(x) = \int_{c-i\infty}^{c+i\infty} \{x^{2t-1} \operatorname{cosec}(\pi t)/\Gamma(2t)\}dt$$

which is discussed with the aid of two different applications of the formula of Stirling. This integral is used to obtain the formula

$$\int_0^{\pi/2} \sin(x \tan a)da = (\pi/4i)\int_{c-i\infty}^{c+i\infty} \{x^{2t-1}\operatorname{cosec}^2(\pi t)/\Gamma(2t)\}dt$$

and the sum of the residues at $t=1, 2, 3, \cdots$ provides the series used by Copson and Hardy [E. T. Copson, Proc. Cambridge Philos. Soc. 37, 102–104 (1941); these Rev. 2, 285].

H. Bateman (Pasadena, Calif.).

Differential Equations

✶Kryloff, N. and Bogoliuboff, N. Introduction to Non-Linear Mechanics. Annals of Mathematics Studies, no. 11. Princeton University Press, Princeton, N. J., 1943. iii+105 pp. $1.65.

This monograph is a free translation from the Russian by Solomon Lefschetz. The major portion of the monograph is a condensation of a book by the two authors devoted to a systematic treatment of problems of practical interest. The treatment is so oriented as to be readily available to the engineer or physicist. In fact, rigor is entirely subordinated to the objective of making the material as widely available as possible. Examples of physical systems are given which lead to the type of equation considered in the monograph. Moreover, general statements of methods for solving equations are illustrated by the explicit solution of examples.

Most of the monograph is devoted to the equation

$$\frac{d^2x}{dt^2}+r^2x+\epsilon f\left(x, \frac{dx}{dt}\right)=0 \tag{1}$$

for small ϵ. This equation, it should be observed, can be qualitatively treated in an exhaustive manner by the methods of Poincaré-Bendixon. This approach, however, is not given in the monograph which deals with perturbation methods exclusively. This latter procedure leads to explicit, though approximate, solutions and can be extended in various directions where other procedures fail. Several procedures for the approximate solution of (1), when ϵ is small, are given, each with an interesting physical interpretation. These various interpretations are finally shown to lead to the same results as the usual perturbation procedure familiar to the mathematician. "Short cut" devices which are very natural from the physical point of view are presented. These devices rapidly yield approximate solutions of (1). In the final sections of the monograph the system

$$m\frac{d^2x}{dt^2}+bx=\epsilon f\left(t, x, \frac{dx}{dt}\right) \tag{2}$$

is considered, where $f(t, x, dx/dt)$ can be represented as the sum of sinusoids in t with coefficients polynomials in x and dx/dt. This equation is quite involved and unlike (1) has not so far yielded to any exhaustive treatment. However, for small ϵ the authors, in a monograph from which Lefschetz presents a few introductory sections, have treated certain cases of (2) rigorously. Lefschetz has performed a considerable service in making this work of Kryloff and Bogoliuboff available now.

N. Levinson (Cambridge, Mass.).

Lefschetz, S. Existence of periodic solutions for certain differential equations. Proc. Nat. Acad. Sci. U. S. A. 29, 29–32 (1943). [MF 7679]

The author proves the existence of at least one periodic solution of period T for the equation

$$\ddot{x}+g'(x)\dot{x}+f(x)=e(t), \tag{*}$$

where $e(t)$ has the period T. He assumes that $e'(t)$, $f'(x)$ and $g'(x)$ exist for all values of t and x, that $f(x)/x\to+\infty$ with $|x|$ and that there exist constants b and B greater than zero and such that $|g(x)-bf(x)|\leqq B|x|$. The author introduces the variable $y=\dot{x}+g(x)$. He replaces (*) by the pair of equations $\dot{x}=y-g(x)$ and $\dot{y}=e(t)-f(x)$. This pair of equations has a solution $(x(t), y(t))$ with $x(0)=x_0$ and $y(0)=y_0$, where (x_0, y_0) are the coordinates of any point P_0. If $(x(T), y(T))$ be denoted by P_1 then we have a transformation which takes any point P_0 of the (x, y) plane into a point P_1. Clearly the existence of a periodic solution of the differential equation is equivalent to the existence of a fixed point for this transformation. The author shows that there is an ellipse which is mapped into its interior by this transformation and therefore, by Brouwer's famous theorem, there is a fixed point.

N. Levinson (Cambridge, Mass.).

Bulgakov, B. V. On the application of Poincaré's method to free pseudo-linear oscillatory systems. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 6, 263–280 (1942). (Russian. English summary) [MF 7581]

The author considers the system

$$\sum_{k=1}^{n} f_{jk}(D, \mu)x_k=\mu\phi_j(x_1, x_2, \cdots, x_n, \mu), \quad j=1, 2, \cdots, n, \tag{*}$$

where D is the differential operator d/dt and μ is small; $f_{jk}(D, \mu)$ are polynomials in D and in practice are usually of the second degree; f_{jk} and ϕ_j are continuous in all their variables. With $\mu=0$ the system (*) is linear with constant coefficients and therefore can be solved quite explicitly. Because of this the author is also able to handle the equation for small μ, by methods due to Poincaré, in a very explicit manner. This he does in the case where $\Delta(D)$, the determinant of (*) with $\mu=0$, has a pair of roots of the form

$\pm i\omega$, and where $\Delta(D)/(D^2+\omega^2)$ has no roots of the form $\pm iL\omega$, where L is any integer.

The author is able to express the amplitude of the oscillation in the nonlinear case, associated with the roots $\pm i\omega$ of the linear case, by means of an equation. This equation is set up explicitly in terms of the determinant $\Delta(D, \mu)$ of the left side of (*), the minors of this determinant and the functions ϕ_j. This equation for amplitude can be solved readily by a graphical procedure. The author applies his method to determine the self-excited oscillations of what he calls follow-up systems, but what here are termed a servo-mechanism. *N. Levinson* (Cambridge, Mass.).

Butenin, N. Maintained vibrating systems of gyroscopic forces. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 327–346 (1942). (Russian. English summary) [MF 7585]

The author studies maintained vibrating systems of gyroscopic forces which deviate only slightly from the linear Hamiltonian forces. As an example he discusses the monorail carriage and a ship fitted with a gyroscopic anti-rolling device. His method is similar to the treatment given by Andronoff for a vacuum tube system in that, by use of approximations, the problem is reduced to considering the solutions of a first order system by methods of Poincaré-Bendixon. Following van der Pol, the system

$$\ddot{x}-\lambda_1\dot{y}-n_1^2x=\mu f(x, \dot{x}, y, \dot{y}),$$
$$\ddot{y}+\lambda_2\dot{x}-n_2^2y=\mu g(x, \dot{x}, y, \dot{y}),$$

for small μ, is assumed to have solutions of the form $x=a\sin\omega_1t+b\sin\omega_2t$, $y=ak_1\cos\omega_1t+bk_2\cos\omega_2t+a_1\sin\omega_1t+b_1\sin\omega_2t$. Certain physically reasonable assumptions lead to the first order system (*) $da/dt=P(a, b)$, $db/dt=Q(a, b)$. These equations are studied qualitatively, in the large, in the (a,b) plane. As the physical parameters are varied, P and Q vary, but the qualitative picture usually remains the same over an interval of values of the parameters and changes only when one passes from one such interval to another. The author follows the usual procedure of presenting in a schematic diagram the several discrete intervals in which the parameters can vary and in each of which the same qualitative picture of the solutions of (*) in the (a,b) plane prevails. *N. Levinson* (Cambridge, Mass.).

Lepage, Th. Sur les champs géodésiques des intégrales multiples. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **27**, 27–46 (1941). [MF 6901]

The writer begins by discussing the algebra of alternating differential forms which are sums of terms of the form

$$A(x_1, \cdots, x_n, t_1, \cdots, t_\nu)dx_{i_1}\cdots dx_{i_p}dt_{j_1}\cdots dt_{j_q},$$

in which the A are holomorphic and $p+q=$constant, the degree of the form. The addition of forms is commutative and associative, and multiplication is associative and distributive but not commutative; instead we have $dx_idx_j=-dx_jdx_i$, $dx_idt_j=-dt_jdx_i$, $dt_idt_j=-dt_jdt_i$. By the differential of a form

$$\sum A_{i,\cdots,j;\,\alpha,\cdots,\beta}dx_i\cdots dx_jdt_\alpha\cdots dt_\beta,$$

we mean the form

$$\sum dA_{i,\cdots,j;\,\alpha,\cdots,\beta}dx_i\cdots dx_jdt_\alpha\cdots dt_\beta$$

(of one higher degree), where this means to form the ordinary differentials of the A and carry out the indicated operations with due regard to the order of multiplications. A form Ω is said to be integrable if there is a form Π such that $d\Pi=\Omega$, a necessary and sufficient condition for this being that $d\Omega=0$ (if we allow multiple valued functions A or restrict ourselves to simple pieces of manifolds). For simple manifolds, we have $\int_{V^{k+1}}\Omega=\int_{V^k}\Pi$, if $d\Pi=\Omega$; Π is of degree k and V^k is the boundary of V^{k+1}.

Now, suppose $L(t_1, \cdots, t_\mu, x_1, \cdots, x_n, p_{11}, \cdots, p_{n\mu})$ is holomorphic in its arguments and consider the totality of forms Ω (of degree μ) with $\Omega\equiv Ldt_1\cdots dt_\mu$, $d\Omega\equiv 0 \bmod \omega_i$, where $\omega_i=dx_i-p_{i\alpha}dt_\alpha$; naturally, we allow the coefficients in Ω to be functions of the extra variables $p_{i\alpha}$. If a matrix of functions $\psi_{i\alpha}(x, t)$ is such that one of the forms Ω above becomes integrable when the $p_{i\alpha}$ are replaced by these functions, the matrix is said to form a geodesic field for the function L. The writer then shows that any matrix $\psi_{i\alpha}(x, t)$, such that the differential system $dx_i-\psi_{i\alpha}dt_\alpha=0$ is completely integrable and the Euler expressions

$$\frac{\partial L}{\partial x_i}-\frac{d}{dt_\alpha}\left(\frac{\partial L}{\partial p_{i\alpha}}\right), \qquad p_{i\alpha}=\psi_{i\alpha}(x, t),$$

are all zero, forms a geodesic field for the function L. The nature of the class of Ω above and procedures for picking out corresponding forms Ω from this class are discussed. *C. B. Morrey, Jr.* (Aberdeen, Md.).

Saltykow, N. Problèmes modernes d'intégration des équations aux dérivées partielles du premier ordre à une fonction inconnue. Bull. Soc. Math. France **68**, 134–157 (1940). [MF 6812]

This paper is a resumé of the lectures delivered by the author at Paris and Strasbourg in 1937. It is expository in nature and is primarily devoted to the solution of the equation $F(x_1, \cdots, x_n, p_1, \cdots, p_n)=0$ by means of the integrable function group of the characteristics. All of the material may be found in the author's "Méthodes modernes d'intégration, etc. [Mémor. Sci. Math., no. 70, Gauthier-Villars, Paris, 1935] and most of it also in Engel-Faber, "Die Liesche Theorie der partiellen Differentialgleichungen erster Ordnung [Teubner, Leipzig, 1932]. *M. S. Knebelman* (Pullman, Wash.).

Gevrey, Maurice. Sur un procédé de résolution, dans le plan, du problème aux limites linéaires le plus général relatif aux équations intégrodifférentielles du type elliptique. C. R. Acad. Sci. Paris **214**, 206–208 (1942). [MF 7844]

In a previous paper [C. R. Acad. Sci. Paris **213**, 635 (1941)] the author used quasi-functions of Green to solve the mixed boundary value problem for the plane region D bounded by the curve C:

$$F_Pu\equiv\mathfrak{E}u+\sum b_i\frac{\partial u}{\partial x_i}+cu+\int_D H_P{}^Qu_Qd\omega_Q+\int_C K_P{}^Mu_Mds_M=f,$$

with $\sum h_i(M)\partial u/\partial x_i+K(M)u=L(M)$ as boundary condition on C. Here $\mathfrak{E}$ is the author's generalization of the elliptic operator

$$\sum a_{ij}\partial^2/\partial x_i\partial x_j.$$

In the present paper, instead of quasi-functions of Green, the same problem is solved using elementary (also quasi-elementary) solutions of $F_Pu=0$. The method of the present paper permits the problem to be solved under more general hypotheses, and is also such that it can be applied to a system of n integrodifferential equations in n unknown functions. *F. G. Dressel* (Durham, N. C.).

Sokolnikoff, I. S. and Specht, R. D. Two-dimensional boundary value problems in potential theory. J. Appl. Phys. 14, 91–95 (1943).

The problem of Dirichlet for two dimensional regions has been treated by many writers. When the region can be mapped conformally within or without a unit circle the formula of Schwartz reduces the problem to a quadrature. The present paper presents the details of the mapping method for the torsion problem of a right prism giving explicit results for the complex torsion function and the twisting moment. It is shown that the method developed by R. M. Morris [Math. Ann. 116, 374–400 (1939)] in series form is an immediate result. In particular, the solution of the torsion problem for a prism whose cross section is the inverse of an ellipse with respect to its center, given by Higgins [J. Appl. Phys. 13, 457–459 (1942); these Rev. 3, 288], is obtained in a very direct manner by the use of the powerful residue theorems of analytic functions. It is the opinion of the reviewer that the method of contour integrals offers advantages not inherently present in the series expansion method. *D. L. Holl* (Ames, Iowa).

Biben, Georges. Sur la généralisation de la méthode de Schwarz. C. R. Acad. Sci. Paris 214, 350–352 (1942). [MF 7853]

The author has in preparation a paper on the application of Schwarz's method to elliptic equations of the type $\Delta_2\psi+P\psi=0$, $P>0$. If $0\leqq t\leqq 1/c$, where c is the constant of Schwarz for the equation, and ψ is a solution of $\Delta_2\psi+tP\psi=0$ which vanishes on the boundary of the domain D, then the following theorem is proved in this note:

$$\int_D \Delta_1\psi dT-t\int_D P\psi^2dT\geqq 0.$$

J. W. Green (Rochester, N. Y.).

Wolf, František. On majorants of subharmonic and analytic functions. Bull. Amer. Math. Soc. 48, 925–932 (1942). [MF 7514]

This paper represents a new approach to a certain group of problems connected with majorants of subharmonic functions. The following result, from which can be deduced results of Levinson and Sjöberg, and also a generalization of the Phragmén-Lindelöf theorem previously given by the author [J. London Math. Soc. 14, 208–216 (1939); these Rev. 1, 48], is established: If $\alpha(x,y)$ is subharmonic in $R[|x|<c, |y|<d]$ and if $\alpha(x,y)<e^{\psi(x)}$, $\psi(x)\subset L$, $x\subset(-c,c)$, $\psi(-c)<\infty$, $\psi(c)<\infty$, then for any δ, such that $0<\delta<d$, there is an upper bound C for $\alpha(x,y)$ in $D_0[|x|<c, |y|<d-\delta]$ dependent only on δ and $\psi(x)$, but independent of the particular $\alpha(x,y)$. A typical application is the following corollary: Let $f(z)$ be a function, analytic in $R[|x|<c, |y|<d]$, such that $|f(z)|\leqq M(x)$. If $\int_{-c}^{c}\log^+\log^+ M(x)dx<\infty$, then for every domain D_0 completely interior to R there exists a φ depending only on D_0 and $M(x)$, such that $|f(z)|\leqq\varphi$ for $z\subset D_0$. *E. F. Beckenbach* (Austin, Tex.).

Infeld, L. A generalization of the factorization method for solving eigenvalue problems. Trans. Roy. Soc. Canada. Sect. III. (3) 36, 7–18 (1942). [MF 7879]

The generalization of the factorization method [for the original communication see Phys. Rev. (2) 59, 737–747 (1941); these Rev. 2, 364] is twofold: (1) an additional general term is introduced into the two complimentary factorized equations, with opposite signs; (2) the eigenvalues associated with each ladder are assumed to be, in general, different from each other. The method, which was formerly restricted to degenerate cases, becomes thus applicable without restriction provided only that factorization is possible. It is shown that equations which do not lend themselves "naturally" to factorization may be "artifically" factorized by introduction of additional terms involving a parameter μ, the original equations being reobtained for $\mu\rightarrow m$ (=second eigenvalue parameter). Artificial factorization cannot be reduced to routine as can the solution of the eigenvalue problem once factorization has been effected, but rather depends on dexterity. The application of the generalized method is demonstrated, first, on the Gegenbauer equation which represents a very simple nondegenerate factorizable case; second, on step-by-step solutions of the perturbed radial Kepler problem of quantum mechanics. Here, the zero and first order solutions are naturally factorizable, but the second order step requires artificial factorization. The same is true in the treatment of the Stark effect by factorization. *H. G. Baerwald.*

Sen, N. R. A note on meson wave. Bull. Calcutta Math. Soc. 34, 61–65 (1942). [MF 7530]

The meson wave equation is

$$\nabla^2\phi-c^{-2}\phi_{tt}-k^2\phi=-4\pi\sigma(x,y,z,t).$$

A formal Beltrami transformation yields

$$\phi_t|_{x,y,z=0}=\int\int\int[\sigma]_-^+\pm(k/2\pi c)[\partial\phi/\partial t]_-^+(e^{-kr}/r)dxdydz,$$

where $[\ ]^+$ is the retarded, and $[\ ]_-$ the advanced, value. This is not a solution, of course, but is of interest perhaps in relating the effect at the origin to the retarded (advanced) values traveling with the uniform velocity c.

D. G. Bourgin (Urbana, Ill.).

Jaeger, J. C. Heat flow in the region bounded internally by a circular cylinder. Proc. Roy. Soc. Edinburgh. Sect. A. 61, 223–228 (1942). [MF 7522]

This paper gives some numerical results for the cooling of a region bounded internally, with constant initial temperature and various types of boundary conditions at the surface. Such problems are of importance in connection with the cooling of mines, etc. A short table of values of an integral occurring in the solution were published by the author and R. Clarke [same Proc. 61, 229–230 (1942); these Rev. 4, 148]. *A. E. Heins* (Cambridge, Mass.).

Kostitzin, Vladimir A. Sur l'équation de la chaleur dans le cas d'une sphère stratifiée avec des sources distribuées sur les surfaces de discontinuité. C. R. Acad. Sci. Paris 214, 461–464 (1942). [MF 7860]

Let S_1 be a sphere of radius r_1 and S_2 be a concentric sphere of radius r_2 $(r_1<r_2)$, and consider the heat equations

$$\partial u/\partial t=k_i\nabla^2u,\qquad i=1,2. \tag{1}$$

For $i=1$ the equation is for the sphere S_1, and for $i=2$ for the region S_2-S_1; the value of k_i depends on the heat properties of the region under consideration. The author proposes to find a solution of (1) in S_1 and in S_2-S_1, satisfying the initial condition $u(r,\varphi,\psi,0)=w(r,\varphi,\psi)$, r,φ,ψ spherical coördinates, and also satisfying boundary conditions depending on t on the surface of S_1 and on the surface of S_2. The author represents the function u by a series of orthogonal functions, but the reviewer could not see that it satisfied the differential equations (1).

F. G. Dressel (Durham, N. C.).

Kostitzin, Vladimir A. Sur l'équation généralisée de la chaleur dans le cas d'une sphère. C. R. Acad. Sci. Paris 214, 47–49 (1942). [MF 7831]

Write the heat equation

(1) $$\nabla^2 u = L(r)\partial u/\partial t$$

in spherical coordinates (r, φ, ψ), and let $L(r)$ be a continuous nonvanishing function of r. The author considers the problem of finding a solution of (1) in a sphere of radius R and satisfying the conditions

(2) $$u(r, \varphi, \psi, 0) = w(r, \varphi, \psi),$$
$$Eu(R, \varphi, \psi, t) = r\partial u/\partial r|_{r=R} + \Psi(\varphi, \psi, t).$$

The reviewer could not see that the representation as printed for u satisfied (1) or (2). *F. G. Dressel.*

Mattioli, G. D. Theory of heat transfer in smooth and rough pipes. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1037, 22 pp. (1942). [MF 7963]

Translated from Forschung auf dem Gebiete des Ingenieurwesens 11, (1940).

Archibald, W. J. The integration of the differential equation of the ultracentrifuge. Ann. New York Acad. Sci. 43, 211–227 (1942). [MF 7465]

The concentration $c(r, t)$ of a solution subjected to the action of a centrifugal force satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(ar\frac{\partial c}{\partial r} - br^2 c\right) = \frac{\partial c}{\partial t},$$

where r is the distance from the axis of rotation, t the time, a and b are constants depending on the angular velocity of rotation on the coefficient of diffusion and on the sedimentation velocity. In the application of the ultracentrifuge the condition arises: $c(r, 0) = \text{const.}$ for $r_1 \leqq r \leqq r_2$ and $\partial c/\partial r = brc$ for $r = r_1$, $r = r_2$. Putting $c(r, t) = \sum R(r)T(t)$, the determination of c is reduced to the solution of the differential equation of the confluent hypergeometric function ${}_1F_1(\alpha, 1, z)$ with the condition $d_1F_1/dz = {}_1F_1$ for two known values of z. The author gives a numerical solution of this boundary value problem and makes a comparison with an experimental result of Svedberg. Use is made of the well-known theorem on orthogonality of eigenfunctions, although that general theorem is not mentioned. There are many misprints in the paper; for instance, on page 216 t^{n-1} should be changed to t^{-n-1}, $b+n-1$ to $\gamma+n-1$, and in the last line (21) into (22). On page 211 "gravitational" should be changed into "centrifugal." The paper is a republication of the previous results of the author [Phys. Rev. 54, 371–374 (1938) and 53, 746–752 (1938)].

I. Opatowski (Chicago, Ill.).

Functional Equations

Ghermanescu, Michel. Équations fonctionnelles du premier ordre. Mathematica, Timişoara 18, 37–54 (1942). [MF 7423]

The paper deals with functional equations of the form

(1) $$F(\theta_m) = A_1(z)\cdot F(\theta_{m-1}) + \cdots + A_m(z)\cdot F(z) + B(z)$$

for a function $F(z)$, where $A_1(z), \cdots, A_m(z), B(z), \theta(z)$ are given functions, and the $\theta_i(z)$ are defined by $\theta_m(z) = \theta(\theta_{m-1}(z))$, $\theta_1(z) = \theta(z)$. Equations of this type have been considered by C. Popovici [Bull. Sci. Math. (2) 53, 213–224, 232–247 (1929)] in connection with integro-differential equations. They present analogies with linear differential equations. It is proved here that the general solution of (1) can be expressed with the help of $m+1$ particular solutions $F_1(z), \cdots, F_{m+1}(z)$ in the form

$$F(z) = \alpha_1(z)F_1(z) + \cdots + \alpha_{m-1}(z)F_{m+1}(z),$$

where $\sum_{i=1}^{m+1}\alpha_i(z) \equiv 1$, $\alpha_i(\theta(z)) \equiv \alpha_i(z)$.

For first order equations (2) $F(\theta) = A(z)F(z) + B(z)$, the author constructs formal solutions by various iteration schemes. He also gives a geometric theory of such equations. With every solution F of (2) is associated the "integral curve" $y = F(x)$. The equation $x' = \theta(x)$ defines a transformation on each such curve. Then, for example: The fixed points of the transformation are common to all integral curves. For any x_0 the straight lines joining the points with abscissae x_0 and $\theta(x_0)$ on any three integral curves are concurrent. *F. John* (New York, N. Y.).

Bagchi, Hari Das. On a class of functional equations. Sankhyā 5, 71–78 (1940). [MF 7348]

The author obtains general solutions for certain functional equations, assuming that particular solutions are known. The equations discussed are:

(A) $\phi(z+a) = f(z)\phi(z) + g(z)$, (B) $\phi(z)\cdot\phi(a-z) = f(z)$,
(C) $\phi(z) = f(z)\cdot\phi(a-z)$, (D) $\phi(z) = f(z)\cdot\phi(1/z)$,

where a is a given constant and $f(z)$, $g(z)$ are known functions. If $\theta(z)$ is a particular solution for each case, then the general solutions are shown to be respectively:

(A)′ $$\phi(z) = \theta(z) + \omega(z)\lambda(z),$$

where $\omega(z)$ is an arbitrary function of period a, and $\lambda(z)$ is a particular solution of $\phi(z+a) = f(z)(\phi(z) - \theta(z))$.

(B)′ $$\phi(z) = \theta(z)g(z)/g(a-z),$$

where $g(z)$ is arbitrary.

(C)′ $$\phi(z) = \theta(z)F(z, a-z),$$

when $F(x, y)$ is an arbitrary function symmetric in x, y.

(D′) $$\phi(z) = \theta(z)F(z, 1/z),$$

where $F(x, y)$ is an arbitrary function symmetric in x, y. Numerous examples are noted. Thus, under (D), the general solution of $\phi(z) = z^{-\frac{1}{2}}\phi(\frac{1}{z})$ is $\phi(z) = \theta(z)F(z, \frac{1}{z})$, where $\theta(z) = \sum_{n=-\infty}^{\infty} e^{-n^2\pi z}$. The fact that this series is a particular solution is due to Poisson [see Whittaker and Watson, Modern Analysis, Cambridge University Press, Cambridge, England, 1927, p. 124]. *M. A. Basoco* (Lincoln, Neb.).

Azevedo do Amaral, Ignacio M. Quelques questions d'analyse itérative. Anais Acad. Brasil. Ci. 11, 273–330 (1939). Errata, 2 pp. [MF 6567]

This paper is designated as an extension of a previous one [Anais Acad. Brasil. Ci. 9, 331–391 (1937)]. The author is mainly concerned with linear iterative functional equations

(1) $$X_p = Y_1X_{p-1} + \cdots + Y_nX_{p-n},$$

where n is a fixed positive integer, p runs through all whole numbers and X_p is the pth iteration of a function (or

operator) X_0 which, together with the coefficients Y_i, are functions of a single independent variable x. Explicit solutions (2) $X_p=f(X_0, Y_1, \cdots, Y_n; C)$ involving an arbitrary constant C are given in the case of several particular cases of (1), and this leads to the concept of a particular and a general solution of (1). In the previous paper a formal solution of the equation

$$D^{(n)}y=\sum_{i=1}^{n} X_i(x)D^{(n-i)}y+X_{n+1}(x), \tag{3}$$

in which D is a generalized derivative operator defined by

$$\begin{aligned} D(X+Y)&=DX+DY,\\ D(XY)&=X(DY)+Y(DX), \end{aligned} \tag{4}$$

is given in the form y equal to a rational function of quantities $X_{i,j}$, $i=1, \cdots, n+1$; $j=1, \cdots, n$, where the $X_{i,j}$ are recursively determined by an equation similar to (1). The solution of (3) is here extended to the case where D is defined by a slight generalization of (4). The errata list appended is hardly immodest. *A. L. Foster.*

TOPOLOGY

Moore, R. L. Concerning intersecting continua. Proc. Nat. Acad. Sci. U. S. A. **28**, 544–550 (1942). [MF 7644]

A study is made of relationships involving a continuum M, its boundary B, the components of $M-B$, the components of the complements of the closures of the components of $M-B$ and the common part of M and another continuum K. Using the theorem quoted in the following review, it is shown under similar conditions that, if M is such that if D is a component of $M-B$ then $B\cdot\bar{D}$ is connected, and K is a continuum intersecting B, then $M\cdot K$ is a continuum. Also if the continuum K contains B and, for every component D of $M-B$, $K\cdot\bar{D}$ is a continuum, then $M\cdot K$ is a continuum. A large number of results of this same general nature are established. *G. T. Whyburn.*

Moore, R. L. Concerning a continuum and its boundary. Proc. Nat. Acad. Sci. U. S. A. **28**, 550–555 (1942). [MF 7645]

The author investigates relationships between the boundary of a continuum and the components of that continuum minus its boundary in a space satisfying axioms 0 and 1 of his book "Foundations of Point Set Theory" [Amer. Math. Soc. Colloquium Series, v. 13, New York, 1932]. It is shown, for example, that, if B is the boundary of a continuum M, if either M is compact or axiom 2 is satisfied and, for every component D of $M-B$, the common part of B and the boundary of D is connected, then B is connected. Numerous results of a similar nature are established and examples given exhibiting the essential nature of the hypotheses. *G. T. Whyburn* (Charlottesville, Va.).

Moore, R. L. Concerning domains whose boundaries are compact. Proc. Nat. Acad. Sci. U. S. A. **28**, 555–561 (1942). [MF 7646]

In a space satisfying axioms 0–5 of the author's book referred to in the preceding review, the existence of a special kind of covering sequence of domains with compact boundaries for a given compact set is established. A similar conclusion is obtained under the assumption of axioms 0–3 and a modification of 5 due to F. B. Jones. With the aid of the latter it is shown that, in the same kind of space, no domain with a compact boundary has uncountably many components. A number of related results, some dealing with continuous curves, are established. *G. T. Whyburn.*

Ehresmann, Charles. Espaces fibrés de structures comparables. C. R. Acad. Sci. Paris **214**, 144–147 (1942). [MF 7841]

This follows up a previous note by Ehresmann and Feldbau [the same C. R. **212**, 945–948 (1941); these Rev. **3**, 58], and another one [ibid. **213**, 762 (1941)] which has not been available to the reviewer. The author uses a definition of fibre-spaces which is essentially equivalent to H. Whitney's [Proc. Nat. Acad. Sci. U. S. A. **26**, 148–153 (1940); these Rev. **1**, 220], and involves not only the "fibre-space" E and its decomposition into fibres F, all homeomorphic to a given "abstract fibre" F_0, but a definite group G of homeomorphisms of F_0 into itself; if B is the "base-space," deduced from E by identifying all points in every fibre, then E is called a fibre-structured space with base B, fibre F_0, and group G. To every fibre F, in that space, there corresponds a homeomorphism φ of F onto F_0 which is well determined up to a transformation s belonging to G; in other words, the set $H(F)=G\cdot\varphi$, consisting of all homeomorphisms $s[\varphi(x)]$ of F onto F_0, where s runs through all elements of G, is uniquely defined for every F; conversely, if such a set $H(F)$ is given for every F, and satisfies certain simple axioms, then it defines a fibre-structure on E. An automorphism of E is a homeomorphism of E, transforming every fibre F into a fibre F' and the set $H(F)$ into $H(F')$. The author states a lemma which, in what seems to be the most useful case, asserts that, if B is a finite complex, then, to every isotopic deformation of B into itself, there corresponds an isotopic deformation of E into itself which is a one-parameter family of automorphisms of E.

To every fibre-space E, defined as above, is attached a "principal" fibre-space E_0 with the same base-space B, the fibre for E_0 being the group G itself, with the group of transformations in G consisting of all left-hand multiplications with elements of G. The author discusses the following problem P: the fibre-structured space E being as above, is it possible to define, on the same space E and for the same decomposition into fibres F, a fibre-structure corresponding to a subgroup G' of G? (If G' is reduced to the unit-element of G, this is the same as the problem of deciding whether E is isomorphic to the direct product $B\times F_0$). The problem is transformed into an equivalent one, on the possibility of defining a continuous system of representatives for B in a certain fibre-space with base B, having as fibre the "homogeneous space" G/G', consisting of the co-sets of G' in G, with left-hand multiplication by elements of G as (transitive) group of transformations. In particular, if B is an n-dimensional complex, and if the homotopy groups of G/G' up to the dimension $n-1$ vanish, then the problem P always has solutions; if, moreover, the homotopy-group of G/G' for the dimension n also vanishes, then all those solutions are equivalent in a suitable sense. The latter is the case, for instance, if G is a non-compact semi-simple Lie-group, and G' a maximal compact subgroup of G (in which case G/G' is a "symmetric Riemannian space" according to E. Cartan). *A. Weil* (Bethlehem, Pa.).

Pontrjagin, L. Characteristic cycles on manifolds. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 34–37 (1942). [MF 7610]

Let E^{k+l} be Euclidean $(k+l)$-space, and $H(k, l)$ the space whose points are the oriented k-planes through the origin of E^{k+l}. If f is a map of an orientable differentiable manifold M^k of dimension k into E^{k+l} with Jacobian of maximum rank at every point, there is an associated map T of M^k in $H(k, l)$; $T(x)$ is the oriented k-plane through the origin parallel to the tangent plane at $f(x)$. If $l \geqq k+2$, it is observed that the homotopy class of T is independent of f, so that the homomorphisms of the homology groups of M^k induced by T are topological invariants. These homomorphisms are completely determined by certain cycles of M^k called characteristic cycles. These are the inverse images in M^k of the intersections of $T(M^k)$ with the cycles of bases in $H(k, l)$. The main portion of the paper is devoted to the construction of independent homology bases in $H(k, l)$ with integer coefficients. As an application, it is shown that, if M^k is mapped into E^s, the set of points where the Jacobian has rank not greater than r is a characteristic cycle. A method is given for computing its homology class. It is also observed that the characteristic cycles include all the characteristic cycles of Stiefel [Comment. Math. Helv. **8**, 3–51 (1936)], and methods of computing the latter are given. The author appears to be unaware of the close relation of his work to that of Whitney on sphere-bundles [Lectures in Topology, University of Michigan Press, Ann Arbor, Mich, 1941, pp. 101–141; these Rev. **3**, 133].

N. E. Steenrod (Ann Arbor, Mich.).

Hall, Dick Wick. A partial solution of a problem of J. R. Kline. Duke Math. J. **9**, 893–901 (1942). [MF 7939]

It is shown that in order for a locally connected continuum M to be homeomorphic with a 2-sphere it is necessary and sufficient that (a) no two points separate M, (b) every simple closed curve J of M separate M, and (c) for any simple closed curve J of M the number of components of $J-M$ be finite. The question remains unsettled as to whether this theorem remains true when (c) is omitted.

G. T. Whyburn (Charlottesville, Va.).

Cartan, Hélène. Sur une caractérisation topologique de la circonférence. C. R. Acad. Sci. Paris **214**, 23–25 (1942). [MF 7833]

The topological characterization of simple closed curves with which this note is concerned is to the effect that a separable connected Hausdorff space E is homeomorphic with a circumference if and only if, for every $x, y \epsilon E, x \neq y$, (1) $E-(x)$ is connected, (2) $E-(x)-(y)$ is not connected and (3) E is locally connected or compact. The author mentions previous characterizations by Janiszewski and Fan but not the result of Moore (a continuum is a simple closed curve if and only if it is disconnected by the omission of any two of its points), which is closely related to the characterization of the note. The option appearing in condition (3) is noteworthy. If E is supposed compact, then, by Moore's theorem, E is a simple closed curve and hence locally connected. Condition (1) is superfluous. On the other hand, both (1) and (2) are needed to show that local connectivity implies compactness.

L. M. Blumenthal.

Wallace, A. D. Monotone transformations. Duke Math. J. **9**, 487–506 (1942). [MF 7332]

This paper studies the class of transformations which are monotone relative to a family of sets. Since proper specializations of the family of sets will make the transformations either monotone or nonalternating, the paper can be considered as another contribution to the generalizations of these types of transformations. The spaces under consideration are, in general, nonmetric, being merely topological spaces in which closed sets are defined and have the usual properties. Consequently the classical theorems needed here regarding continua must be proven, and this the author does in the early part of the paper. He proves, for example, that every bicompact and connected T_1 space containing at least two points contains at least two noncutpoints of itself. A cyclic element theory for bicompact Hausdorff spaces is worked out in the second section of the paper, and it is here that the author obtains the rather surprising result that, if p is a point of such a space which is neither a cutpoint nor an endpoint, then the space must contain a point q which is conjugate to p. Among the results on transformations obtained in the concluding section of the paper is an extension of the theorem of Schweigert [Abstract in Bull. Amer. Math. Soc. **44**, 636 (1938)] to the effect that A-sets are invariant under nonalternating transformations.

D. W. Hall (College Park, Md.).

Menger, Karl. What is dimension? Amer. Math. Monthly **50**, 2–7 (1943). [MF 7940]

An expository article.

NUMERICAL AND GRAPHICAL METHODS

***Davis, Dale S. Empirical Equations and Nomography.** McGraw-Hill, New York, 1943. ix+200 pp. $2.50.

This book contains a brief (about 40 pages of theory) and quite elementary account of empirical equations and nomography, copiously illustrated with worked out examples and supplemented by many problems. The book is, in fact, primarily an excellent collection of problems and examples. "Illustrations are drawn from recent mechanical, civil and chemical engineering literature, and the problems deal with charts now in actual daily use in the dairy, paper, textile, refrigeration, heavy chemical, petroleum and other industries." The reader is impressed with the enormous variety of these applications. For this reason the book will doubtless appeal to the industrialist and engineer who wishes an introduction into the subject with an indication of the present scope of its applications, but with little required mathematical background.

From a mathematical view the methods in the book appear unnecessarily special. Statistical methods are not mentioned. Curve fitting by Fourier series or other infinite series is omitted. The general equation for a three scale nomograph is not even given. Also, the reviewer questions the pedagogical desirability of stating every illustration and problem in engineering terms. The general principles for constructing nomographs can probably be illustrated most simply by charts for the fundamental processes of calculation, considered abstractly, such as extraction of roots, solution of quadratic equations, laws of sines and cosines, etc., none of which are considered in this book. The physicist might also deplore the continual use of equations which are dimensionally incorrect.

P. W. Ketchum.

Lowan, Arnold N., Salzer, Herbert E. and Hillman, Abraham. A table of coefficients for numerical differentiation. Bull. Amer. Math. Soc. **48**, 920–924 (1942). [MF 7513]

"The . . . table lists the coefficients $A_{m,s}$ for $m=1, 2, \cdots, 20$ and $s=m, \cdots, 20$ in Markoff's formula for the mth derivative in terms of advancing differences, namely,

$$\omega^m f^{(m)}(x)=\sum_{s=m}^{n-1}(-1)^{m+s}A_{m,s}\Delta^s f(x)+(-1)^{m+n}\omega^n A_{m,n}f^{(n)}(\xi).$$

In this formula ω is the tabular interval and

$$A_{m,s}=(-1)^{m+s}mB_{s-m}^{(s)}/s(s-m)!$$

and B_{s-m} is the $(s-m)$th Bernoulli number of the sth order." [Quotation from the paper.] *W. E. Milne.*

Jaeger, J. C. and Clarke, Martha. A short table of $\int_0^\infty(e^{-xu^2}/(J_0^2(u)+Y_0^2(u)))(du/u)$. Proc. Roy. Soc. Edinburgh. Sect. A. **61**, 229–230 (1942). [MF 7523]

This note gives a table of values for the integral mentioned in the title. This integral is a special case of an integral appearing in a paper by J. C. Jaeger [the same Proc. **61**, 223–228 (1942); these Rev. **4**, 144]. *A. E. Heins.*

Segal, B. I. Approximate calculation of some hyperelliptic integrals that occur in the design of dams. C. R. (Doklady) Acad. Sci. URSS (N.S.) **35**, 198–202 (1942). [MF 7617]

The author obtains an approximate value for the definite integral

$$J_1=\int_\beta^\gamma[(\zeta^2-1)(\gamma^2-\zeta^2)(\zeta-\alpha)(\zeta-\beta)]^{-\frac{1}{2}}(\zeta-\sigma)d\zeta,$$

where $-1<\sigma<1$, $1<\alpha<\beta<\gamma$ and $\beta-\alpha$ is small. Similar integrals in different ranges of values of ζ are also evaluated. *M. C. Gray* (New York, N. Y.).

Thomas, Maurice. Sur la quadrature approximative d'une courbe. C. R. Acad. Sci. Paris **214**, 654–656 (1942). [MF 7889]

Various approximate expressions are obtained for the integral of a function y from x_p to x_q in terms of equally spaced ordinates. For example, if S_1 denotes the value of the integral as found by the Trapezoidal Rule and S_2 the value of the integral obtained by applying a third degree polynomial arc to each individual interval (that is, a third degree polynomial fitted to the two bounding ordinates and the adjacent ordinates on each side) the author shows that

$$S_2=S_1-\frac{h}{24}(y_{p-1}-y_{p+1}-y_{q-1}+y_{q+1}).$$

Analogous expressions are obtained for the case where fifth degree, seventh degree, etc. polynomials are used instead of third degree polynomials. Formulas are also given based on third degree polynomial arcs joined so as to have continuous tangents. No estimate of the error is given. *W. E. Milne* (Corvallis, Ore.).

Vernotte, Pierre. Formule pour la quadrature empirique d'une fonction expérimentale. C. R. Acad. Sci. Paris **214**, 107–110 (1942). [MF 7838]

The author observes that if one applies the Newton-Cotes formulas (or similar formulas based on polynomial approximation) to the integration of a function given by n equidistant experimentally determined ordinates, the coefficients by which the ordinates are multiplied vary widely in magnitude and thus the experimental errors in the ordinates receive widely different weights. He therefore proposes the formula

$$\int_{x_1}^{x_n}ydx=\frac{h(n-1)}{2n(n-2)}[(n-2)y_1+2(n-1)(y_2+\cdots+y_{n-1})+(n-2)y_n].$$

This formula is exact if y is a polynomial of degree 3 or less. No estimate of the error is given. [Reviewer's Note. The formula above reduces to Simpson's Rule for $n=3$, to Newton's "three eighths" rule for $n=4$, and to combinations of these and fourth differences for larger values of n]. *W. E. Milne* (Corvallis, Ore.)

Saibel, Edward. A modified treatment of the iterative method. J. Franklin Inst. **235**, 163–166 (1943). [MF 7919]

The iterative method is applied to finding the greatest characteristic value of a symmetric matrix according to a well-known method. The author's modification is an improvement of the comparatively crude method outlined in Frazer, Duncan and Collar, Elementary Matrices [Cambridge University Press, 1938]. However, this modification is not new; it is contained among Aitken's refinements, of which the author seems unaware [Proc. Roy. Soc. Edinburgh. Sect. A. **57**, 269–304 (1937)]. *W. Feller.*

Samuelson, P. A. A method of determining explicitly the coefficients of the characteristic equation. Ann. Math. Statistics **13**, 424–429 (1942). [MF 7872]

The problem is to determine, for a given matrix $|a|$, the coefficients $D(\lambda)=|\lambda I-a|=\lambda^n+p_1\lambda^{n-1}+\cdots+p_n$. After mentioning three known methods and indicating a fourth method suggested by E. Bright Wilson, Jr., a new method is explained. The author starts from the reduction of a differential equation $X^{(n)}+p_1X^{(n-1)}+\cdots+p_{n-1}X'+p_nX=0$ to a normal system containing n first order equations: $X_i'=\sum b_{ij}X_j$, $i, j=1, \cdots, n$, where $b_{ij}=0$, $j\neq i+1$; $b_{i,i+1}=1$ for $i=1, \cdots, n-1$ and $b_{nj}=-p_{n+1-j}$. Then $|b_{ij}|$ is the companion matrix of the λ-polynomial.

Conversely, given a normal system, the required coefficients are given by the last row of the companion matrix. Now assume given a matrix $|a_{ij}|$ which is not of the standard $|b_{ij}|$ form. There exists a nonsingular matrix C such that $C^{-1}|a_{ij}|C=|b_{ij}|$ and $|a_{ij}|$, $|b_{ij}|$ have the same characteristic equation. An algebraic method of finding C is examined. While the formal set-up is formidable (n^2 linear equations in n^2+n variables), the author explains essential short-cuts.

The last paragraph contains a comparison between earlier methods and the new method. Measuring the efficiency in terms of the number of multiplicative operations involved, it is stated that for $n<6$ the new method seems to be more powerful than the other known methods. *A. J. Kempner* (Boulder, Colo.).

Morris, J. and Head, J. W. Lagrangian frequency equations. An "escalator" method for numerical solution. Aircraft Engrg. **14**, 312–314, 316 (1942). [MF 7560]

The authors describe a new inductive method (not using iterations) to find the characteristic values and characteristic vectors satisfying an equation of the form (*) $AX=\lambda BX$,

where A and B are $n\times n$ matrices, X a vector, λ a scalar. Let A^* and B^* be the $(n-1)\times(n-1)$ matrices obtained by deleting the last rows and columns of A and B, respectively. Supposing that the $n-1$ characteristic vectors and values corresponding to $A^*X^*=\lambda B^*X^*$ are known, the authors show how to compute those corresponding to (*). The method involves the solution, for all n roots, of an equation of nth degree, but no computation of determinants. Starting, then, with the 2×2 matrices in the upper left corners of A and B one works by steps up to the complete solution. In all, one requires the complete solution of $n-1$ equations of degrees 2, 3, $\cdots$, n, respectively. An application to the solution of normal equations is also given. In an Appendix, G. Temple gives a proof, in matrix form, of the validity of the method. [For a previous article by the first of the authors cf. Aircraft Engrg. 14, 108–110 (1942); these Rev. 4, 90.] *W. Feller* (Providence, R. I.).

von Sanden, H. Zur Berechnung des kleinsten Eigenwerts von $y''+\lambda p(x)y=0$. Z. Angew. Math. Mech. 21, 381–382 (1941). [MF 7671]

Let $0\leqq x\leqq 1$, $p(x)>0$ and $y(0)=y(1)=0$. Let $y_1(x)$ be an arbitrary function which is positive for $0<x<1$ and vanishes for $x=0$, 1. Define y_{n+1} by repeated integrations so that $y''_{n+1}=-p(x)y_n$ and $y_{n+1}(0)=y_{n+1}(1)=0$. Then the smallest characteristic value λ_1 lies between the maximum and the minimum of y_n/y_{n+1}, and this ratio tends to λ_1. This fact is the basis for a widely used method of practical computation of λ_1. The author gives a simple proof using the expansion of y_1 into a series of characteristic functions.
W. Feller (Providence, R. I.).

Collatz, Lothar und Zurmühl, Rudolf. Beiträge zu den Interpolationsverfahren der numerischen Integration von Differentialgleichungen 1. und 2. Ordnung. Z. Angew. Math. Mech. 22, 42–55 (1942). [MF 7669]

The author describes a modification of J. C. Adams' method, based on an iterative process. Let $y'=f(x, y)$. Using Stirling's interpolation formula one has, in customary notation, (*) $y_{n+1}=y_{n-1}+h(2f_n+\frac{1}{3}\Delta^2 f_{n-1})$. In actual practice $\Delta^2 f_{n-1}$ is not known. The author proposes to start with a rough approximation to the true value of $\Delta^2 f_{n-1}$ obtained by an extrapolation from values $\Delta^2 f_{n-2}$ already computed, etc. With this approximation we compute y_{n+1} according to (*); this value for y_{n+1} yields an improved value for $\Delta^2 f_{n-1}$, and so on. It is shown how the convergence can be accelerated. The best choice of h is discussed and the error involved is (favorably) compared with that incurred in Adams' method. The new procedure is then generalized to equations of the form $y''=f(x, y, y')$. *W. Feller.*

Pugachev, V. S. Problem of exterior ballistics of bombs. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 6, 281–286 (1942). (Russian. English summary) [MF 7582]

The author is concerned with the exterior ballistics of a particle in a motionless atmosphere neglecting curvature and rotation of the earth. The variations of the air density and of the temperature are taken into account. It is simple to arrive at a system of two differential equations for the two components of the velocity-vector as a function of the height y. After integrating this system the abscissa x and the time t are obtained by simple quadratures. The author describes two methods of approximate integration of the equations assuming the initial position to be at the summit of the trajectory. The first method can be described as one of successive approximations. In the second method a parameter is introduced and the integrals are expanded into a power series in this parameter. No numerical examples are given. *W. Feller* (Providence, R. I.).

Pugachev, V. S. Notes on exterior ballistics of projectiles and bombs. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 6, 347–368 (1942). (Russian. English summary) [MF 7586]

This paper is devoted to the integration of the flight equations of a projectile of finite dimensions. As in the paper reviewed above, the air is assumed at rest and the earth flat. However, the axis of the projectile is not tangent to the trajectory. The projectile itself is spinning, finned or both. Using convenient coordinates attached to the projectile, the author separates an oscillatory component of motion from a nonoscillatory part; the latter can be determined by the usual methods of exterior ballistics. The author describes methods of approximate integration of the differential equations of the oscillating component and studies the stability of the motion.

The English summary is four pages long and refers to the numbered equations of the Russian text. *W. Feller.*

Pugachov, V. S. On the approximate solution of the general problem of exterior ballistics. Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 5, 263–266 (1941). (Russian. English summary) [MF 7710]

The author describes a modification of Siacci's method of approximate integration of the flight equations of a particle. The new procedure apparently works under somewhat more general conditions. No numerical examples are given. The English summary, together with the equations of the text, should suffice for a reader not acquainted with Russian.
W. Feller (Providence, R. I.).

Waschakidze, D. Über die numerische Lösung der biharmonischen Gleichung. Trav. Inst. Math. Tbilissi [Trudy Tbiliss. Mat. Inst.] 9, 61–73 (1941). (Russian. German summary) [MF 7379]

In this paper the author discusses two methods for (actual) solution of the boundary value problem of the equation $\Delta\Delta U=0$. He assumes that ϕ, ϕ_x and ϕ_y are given on the boundary L of the domain and replaces the differential equation $\Delta U=\phi(x, y)$, $\Delta\phi=0$, by a set of corresponding difference equations. He also replaces the boundary curve L by L', a new curve which passes through the vertices of the squares. Using an idea of Mikeladze [Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR] 5, 57–74 (1941); these Rev. 2, 368], he adds new equations, so that the necessary number of equations is obtained. The second method consists of replacing the equation $\Delta\Delta U=0$ by the Richardson difference equation and adding the necessary number of additional equations. Numerical examples of the application of both methods are given. *S. Bergman.*

Aprile, Giuseppe. Funzioni generatrici generalizzate, e loro applicazione ai metodi grafico-numerici di valutazione nel calcolo operatorio funzionale. Pont. Acad. Sci. Acta 6, 139–146 (1942). [MF 7468]

The graphical way of interpreting symbolical equations $W(t)=f(\Delta)V(t)$ may effectively replace the analytical way

when $V(t)$ is hard to use analytically or when it is given graphically. The functional operator $f(\Delta)$ is fully characterized by its effect $G(t)$ on the unitary impulsive function $Fu(t)$ or by its effect $H(t)$ on the unit step $1(t)$. More generally, if $j_n(t)=\Delta^n Fu(t)$, a generalized generating function of order n may be defined by the equation $A_n(t)=f(\Delta)j_n(t)$, and then $G(t)=A_0(t)$, $H(t)=A_{-1}(t)$. The case $n=2$ is of special interest and is illustrated by diagrams. Other generalizations are possible, as the author shows.

H. Bateman (Pasadena, Calif.).

Mokrzycki, G. A. Graphic determination of distance in accelerated airplane motion. J. Aeronaut. Sci. **10**, 36–37 (1943). [MF 7829]

MATHEMATICAL PHYSICS

✶Keller, Ernest G. Mathematics of Modern Engineering. Vol. II. Mathematical Engineering. John Wiley and Sons, Inc., New York, 1942. xii+309 pp. $4.00.

This is the second volume of a book, the first having been written by Robert E. Doherty and the present author (1936). The volume under review falls into three distinct and almost wholly unrelated parts which are entitled (I) Engineering Dynamics and Mechanical Vibrations, (II) Introduction to Tensor Analysis of Stationary Networks and Rotating Electrical Machinery, (III) Nonlinearity in Engineering.

Part (I) contains standard material. Included are a brief treatment of the calculus of variations, Hamilton's principle, the equations of Lagrange with applications to vibration problems, energy methods and the Rayleigh-Ritz method for approximating eigenvalues.

Part II is devoted to an explanation of the methods of Gabriel Kron in electrical engineering. It is not easily read. However, the portion of this section devoted to stationary networks can be understood by those who are already thoroughly familiar with the basic ideas and terminology of electrical engineering. The general idea would seem to be as follows. Any network can be broken up into a finite number of distinct coils, a coil being defined as a closed circuit containing an impedance in series with an electromotive force. The given network can then be described by stating the manner in which the "primitive" coils are connected. In any network it is always possible to choose sets of independent mesh currents in different, but essentially equivalent, ways. Any one set of currents can then be obtained from those assumed in the primitive coils, through Kirchhoff's laws, by means of a linear transformation of matrix C which is called the connection matrix. The elements of C will always be one of the numbers 0, +1 or −1. The differential equations of performance of the network can then be determined by appropriate matrix multiplications with C and with the matrix Z of the impedances in the primitive coils. Tensor analysis is introduced formally by interpreting a set of currents as a vector, the components of which undergo linear transformations. In actual practice, only the matrix notation is used, at least for networks. However, the author makes the statement: "It is emphasized that the formal manipulation is matrix multiplication, but the analysis is not matrix analysis. The analysis is tensor analysis."

The method of analyzing rotating electrical machinery follows the same general plan, but is much more complicated. The analysis of all rotating machines is made to depend upon simple "primitive machines," differences in individual machines being accounted for through differences in the connection of the coils in the windings of the machines. A C-matrix is obtained in terms of which the differential equations of performance of any special machine can be derived by matrix multiplications of a routine character. The general advantages of the Kron procedure for stationary networks are not to be denied. To begin with, the methods make it possible to obtain the differential equations for the network by a purely routine process, an advantage of some importance in complicated cases. If a number of networks are to be analyzed which differ only in a few meshes, much of the analysis made for any one individual case can be taken over easily for the other cases. Also, if two or more complicated networks are connected together, the procedure of Kron makes it possible to write down readily the differential equations for the combined network once the individual networks have been dealt with.

Section (III) of the book introduces the standard methods of approximating the solutions of nonlinear differential equations, with most of the emphasis on the perturbation method and the method of variation of parameters. A number of concrete examples, mostly taken from the field of nonlinear vibrations, are worked out in detail. A short study of elliptic integrals and functions is included in this section. The author recommends that problems like that of the forced oscillation of the pendulum with large amplitudes be attacked by determining first the free vibration through explicit integration using elliptic integrals followed by determination of the forced oscillation through use of the method of variation of parameters.

At the end of each of the three sections an extensive bibliography is given. *J. J. Stoker* (New York, N. Y.).

✶Gardner, Murray F. and Barnes, John L. Transients in Linear Systems. John Wiley and Sons, Inc., New York, 1942. viii+389 pp. $5.00. here v.1 (1942)

This book is the first of a two volume set and has the following aims. It is intended that this text offer a discussion of a certain group of physical problems which may be formulated mathematically as initial or boundary value problems of ordinary linear difference or differential equations. It is further intended to demonstrate how such problems may be solved with the aid of the Laplace transform. On the mathematical side we find a thorough discussion of certain phases of the Laplace transform including order properties of these transforms and the faltung theorem. There is an excellent group of physical problems which are drawn from the theory of partical dynamics and lumped parameter electric circuit theory. The book ends with a series of appendices including such topics as a table of Laplace transforms, comparison of the Laplace and Fourier integral transformation methods, historical notes and a good bibliography. The book should supply a good intermediate course in mathematics for students who are concerned with the particular class of linear problems discussed in this text. In particular, students who are concerned with the study of transients in linear mechanical and electrical systems now have at their disposal a text which will give them a good basic discussion of the physical phases of such problems as well as a method to solve them.

A. E. Heins (Cambridge, Mass.).

Kron, Gabriel. Equivalent circuits for oscillating systems and the Riemann-Christoffel curvature tensor. ~~Elec. Engrg.~~ 62, 25–31 (1943). Trans. Amer. Inst. Elec. Engrs.

Many electromechanical systems involving rotating machinery are so complex that their behavior cannot be computed directly but only by setting up equivalent circuits ("network analyzers"). In investigations of the stability of systems ("hunting" performance), it was frequently found impossible to establish one-to-one correspondences with stationary a.c. networks. It is pointed out that this was invariably due to the fact that, while the Lagrangian equations of the whole system were tensor equations, the equations of small oscillations derived from them by standard methods were not, and therefore the variables involved did not represent physical quantities. The difficulty is overcome by adding self-cancelling arrays which, when combined with the component arrays constituting the hunting equations, will make them covariant. The theory and application of the method is illustrated by several examples, which, however, are only of value for those intimately acquainted with the author's former work, as both notations and procedures are largely unexplained in the present article.

H. G. Baerwald (Cleveland, Ohio).

Higgins, Thomas James. The inductance of tubular conductors of eccentric-annular cross-section. J. Math. Phys. Mass. Inst. Tech. 21, 159–177 (1942). [MF 7759]

The low-frequency formula for the inductance per unit length of two parallel cylindrical conductors, their cross-sections being bounded by eccentrically located circles, is derived for the case in which each conductor, the region surrounding the conductors, and the cylinders enclosed by the conductors have different permeabilities. Errors in special cases treated many years ago by Maxwell and by MacDonald are noted. *J. L. Barnes* (Princeton, N. J.).

Riordan, John and Shannon, C. E. The number of two-terminal series-parallel networks. J. Math. Phys. Mass. Inst. Tech. 21, 83–93 (1942). [MF 7249]

This paper is an elaboration of the work of P. A. MacMahon [Electrician 28, 601–602 (1892)]. The authors give a proof of MacMahon's generating function, and they develop various recurrence relations and schemes of computation with which to extend the table of values of s_n from $n=10$ to $n=30$, where s_n is the number of electrically distinct two-terminal series-parallel networks composed of n elements. An error in MacMahon's tabulated result for $n=10$ is noted. Various approximations to s_n for large values of n are derived, but no true asymptotic formulae are found. *R. M. Foster* (New York, N. Y.).

Searle, G. F. C. The force required to give a small acceleration to a slowly-moving sphere carrying a surface charge of electricity. Philos. Mag. (7) 33, 889–899 (1942). [MF 7789]

This mathematically elementary paper gives a discussion of various matters connected with the well-known formula for the "electromagnetic mass" of a sphere, particularly with regard to the appropriate approximations used in its derivation. Three different methods are given. *D. C. Lewis.*

Jouguet, Marc. Sur les oscillations électromagnétiques naturelles d'une cavité ellipsoidale. C. R. Acad. Sci. Paris 214, 214–215 (1942). [MF 7845]

In an earlier communication [the same C. R. 209, 203–205 (1939)], the author had derived the wave equation for electromagnetic vibrations in cavities bounded by surfaces of revolution, inside perfect conductors; it refers to a scalar function U in terms of two orthogonal coordinates, x_1, x_2, the limiting surface being $x_1=a$. The solutions for the electric and magnetic vectors are in terms of $\partial U/\partial x_1$, $\partial U/\partial x_2$ and satisfy the continuity and boundary conditions; they consist of two types, "electric" and "magnetic" with the components E_φ, H_1 and H_2, and H_φ, E_1, and E_2, respectively (φ=angle of revolution). The ellipsoidal case is considered in particular. It is separable and affords even and odd solutions obtained in well-known ways by expansion; the recursive formulas for the coefficients and the determination of the parameters in terms of definite integrals over the eigenfunctions are presented. *H. G. Baerwald.*

Ginsburg, V. L. On the reflection of an electromagnetic impulse from the Heaviside layer. Acad. Sci. USSR. J. Phys. 6, 167–174 (1942). [MF 7773]

The author considers the deformation of an electromagnetic pulse when reflected from a nonuniformly ionized layer (the Heaviside layer) whose electron concentration varies quadratically or linearly. This problem is a generalization of one considered by Sommerfeld [Ann. Physik 44, 177 (1914)] and Brillouin [Ann. Physik 44, 203 (1914)].

A. E. Heins (Cambridge, Mass.).

Ramachandran, G. N. Reflection of light by a periodically stratified medium. Proc. Indian Acad. Sci., Sect A. 16, 336–348 (1942). [MF 7950]

The complex amplitudes of transmitted and reflected rays in the nth stratum relative to the incident amplitude are found for general complex transmission and reflection coefficients t, r of each stratum, via the 2nd order linear difference equation, the problem being formally very similar to the well-known one of the recurrent electrical structures. The result is specialized for the case considered by Lord Rayleigh, that the periodic refractive index changes in a rectangular wave fashion; the angle of incidence is arbitrary. The result is discussed regarding the sharpness of the primary maxima which is shown to be limited by the reflective power of the stratum, that is, cannot be indefinitely increased by increasing their numbers. It also follows that the intensity distribution of the secondary maxima is not in general symmetrical with respect to the primary ones.

H. G. Baerwald (Cleveland, Ohio).

Roubaud-Valette, Jean. Les équations de Maxwell et l'espace elliptique à trois dimensions. C. R. Acad. Sci. Paris 214, 60–62 (1942). [MF 7836]

In three-dimensional elliptic space there are two types of displacement sending a given point P into a given point Q. The author makes the assumption that the vector potential describing an electromagnetic field in such a space will return unchanged if subjected first to a displacement of one type and then to the reversed displacement of the other type. He derives as a consequence equations analogous to the Maxwell field equations, but involving the radius of curvature of the space. *O. Frink* (State College, Pa.).

Cattermole, Jessie. Relativistic aspect of the stress tensor of the electromagnetic field. Philos. Mag. (7) 33, 674–678 (1942). [MF 7230]

The author denotes by $R_{\alpha\beta}$ and R the Ricci tensor and the scalar curvature of a five dimensional space with the metric

$$\gamma_{\alpha\beta}=g_{\alpha\beta}+\frac{\kappa_\alpha\kappa_\beta}{\kappa_5^2}\gamma_{55} \qquad \begin{matrix}(\alpha,\beta=1,2,\cdots,5)\\(i,j=1,\cdots,4),\end{matrix}$$

where

$$g_{ij}=\delta_{ij},\quad g_{i5}=0$$

k_1, k_2, k_3 are the components of the electromagnetic vector potential, $k_4 = iV/c$, V the scalar potential, $i = \sqrt{-1}$ and c is the velocity of light; k_5 and γ_{55} are constants. The quantity $R_\beta^\alpha - \frac{1}{2}\gamma_\beta^\alpha R$ is computed and it is noted that if Maxwell's equations for five space are satisfied $R_j^i - \frac{1}{2}\gamma_j^i R$ is equal to the electromagnetic stress tensor. It is then "concluded" that $R_\beta^\alpha - \frac{1}{2}\gamma_\beta^\alpha R$ are the components of the stress tensor for an electron (or positron) field. *A. H. Taub.*

Tonnelat, Marie-Antoinette. Théorie de la particule de spin maximum 2. Les tenseurs symétriques du second rang. C. R. Acad. Sci. Paris **214**, 253–256 (1942). [MF 7847]

The author defines three symmetric tensors in this theory and expresses them as functions of quantities appearing in the theory of the particle of maximum spin 1. The three tensors are equivalent in the special case of a plane monochromatic wave. Unlike the theory of the particle of maximum spin 1, it is found here in the case of a particle of total spin 1 that none of the three tensors is purely Maxwellian, and that the analogues of a certain positive definite tensor in the former theory are not positive definite.
A. Schwartz (State College, Pa.).

Nordsieck, A., Lamb, W. E., Jr. and Uhlenbeck, G. E. On the theory of cosmic-ray showers. I. The Furry model and the fluctuation problem. Physica **7**, 344–360 (1940). [MF 7629]

According to the Furry model, an electron moving with energy E through matter may experience either of two processes. First, it may start another electron moving by impact in the same direction and with energy between u and $u+du$, itself continuing in this direction with energy $E-u$ (probability per unit thickness of matter equals $q(E, u)du$). Secondly, it may lose energy by ionization, dropping to the energy interval $(u, u+du)$ (probability per unit thickness equals $p(E, u)du$). It is understood that all electrons produced by the first process may experience these processes in their turn. If now a single electron of energy E_0 is incident normally upon a plate of homogeneous matter, the nature of the emerging shower of electrons is specified by the "master function" giving the probability that, after crossing a thickness x of matter, any given number of electrons emerge in any given energy interval. It is specified less completely by such functions as $P(E_0, N, x)$, the probability that x electrons emerge after thickness x, or $F(E_0, E, x)$, the average number of particles of energy E.

After a detailed and well documented discussion of all these ideas, the authors lay down the integrodifferential equation of conservation

$$(1)\quad (\partial/\partial x)F(E_0, E, x) = -F(E_0, E, x)\int_0^E q(E, u)du - F(E_0, E, x)\int_0^E p(E, u)du + 2\int_E^\infty q(u, E)F(E_0, u, x)du + \int_E^\infty p(u, E)F(E_0, u, x)du.$$

This must be solved for $F(E_0, E, x)$ with the initial condition $F(E_0, E, 0) = \delta(E-E_0)$, $\delta(\xi)$ being the Dirac peak function. This problem is simplified by the following assumptions which have a substantial physical basis. (a) $q(E, u)du$ depends only on the energy lost; accordingly we write $q(E, u) = E^{-1}\chi(u/E)$, $\chi(\xi) = \chi(1-\xi)$, $\int_0^1 \chi(\xi)d\xi = B$. (b) The energy is lost by ionization only in very small steps (namely, continuously), so that $p(E, u) = (\beta/\Delta)\delta(E-u-\Delta)$. Here $\Delta \ll E, u$; β is the average loss of a particle per unit distance, and is assumed independent of E. With these assumptions (1) becomes, after letting $\Delta \to 0$,

$$(2)\quad (\partial/\partial x)F(E_0, E, x) = -BF(E_0, E, x) + 2\int_E^\infty (du/u)\chi(E/u)F(E_0, u, x) + \beta(\partial/\partial E)F(E_0, E, x).$$

In the case of no ionization ($\beta = 0$) this is readily solved by a Mellin transformation. The whole physical problem consists in the solution of (2) when $\beta \neq 0$, and this had proved insurmountable at the time of writing, 1940 [however, cf. the following review].

In the remainder of the paper, (2) is considered with $\beta = 0$, and the effect of ionization is roughly taken into account by the "cut-off method" which consists in replacing the energy interval $(0, E_0)$ by (ϵ, E_0), ϵ being a positive constant to be determined by physical considerations. Extensive calculations based on the method of steepest descent are then employed. Since the authors have more recently superseded their own results [cf. the following review] we omit further details. *B. O. Koopman* (New York, N. Y.).

Scott, W. T. and Uhlenbeck, G. E. On the theory of cosmic-ray showers. II. Further contributions to the fluctuation problem. Phys. Rev. (2) **62**, 497–508 (1942). [MF 7650]

The ideas and methods of the earlier paper [cf. the paper reviewed above] are recapitulated, extended and corrected. In a conversation with one of the authors of the earlier paper, the reviewer has been informed that one of the authors of this paper has in recent unpublished investigations gone much further towards the final mathematical solution of equation (2) [cf. the preceding review], and accordingly he does not go into further detail concerning the computational results in this paper. *B. O. Koopman.*

BIBLIOGRAPHICAL NOTE

Mathematical Tables and Aids to Computation. A Quarterly Journal edited on behalf of the Committee on Mathematical Tables and Aids to Computation by the Chairman, Raymond Clare Archibald.

Volume 1, no. 1 (1943) has appeared. The main part of the issue (pp. 2–23) is devoted to a section "Recent Mathematical Tables." Under this heading, tables published currently, and within the past ten years, are reviewed and this section is to be continued. The issue contains, furthermore, sections entitled "Mathematical Tables—Errata" (pp. 24–26), "Unpublished Mathematical Tables" (pp. 26–27), "Mechanical Aids to Computation" (pp. 27–28), "Notes and Queries" (pp. 29–31).

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